

On the motion of timelike minimal surfaces in the Minkowski space \mathbf{R}^{1+n}

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Abstract

In this paper we are devoted to the study of the motion of the timelike minimal surfaces in the Minkowski space \mathbf{R}^{1+n} . Those surfaces are known as membranes or relativistic strings, and described by a system with n nonlinear wave equations of Born-Infeld type. We construct a global timelike Sobolev regularity torus in \mathbf{R}^{1+n} , which time slice are evolved by a rigid motion. A Lyapunov-Schmidt decomposition reduces this problem to an infinite dimensional bifurcation equation and a range equation. To overcome the higher order derivative perturbation in bifurcation equation and the “small divisor” phenomenon in range equation, a suitable Nash-Moser iteration is constructed.

1 Introduction and Main Results

Our aim in this paper is to study the motion of the action for the world-volume $x = (x^0, x^1, \dots, x^{m-1})$ of the surface Σ moving in \mathcal{M} be given by the $(1+n)$ -dimensional volume swept out in space-time

$$\Upsilon[x] = \int \sqrt{G} ds^0 ds, \quad (1.1)$$

where the surface Σ is n -dimensional manifold with coordinates $s = (s^1, \dots, s^n)$, (\mathcal{M}, g) is m -dimensional Lorentz manifold and G is given by

$$G = |\det(G_{\alpha\beta})|, \text{ and the induced world volume metric } G_{\alpha\beta} = x_{,\alpha}^\mu x_{,\beta}^\nu g_{\mu\nu}(x),$$

where the index $\alpha, \beta, \dots = 0, \dots, n$ and $\mu, \nu = 0, \dots, m-1$. Here and in the sequel, we use the Einstein summation convention.

The Euler-Lagrange equations for the action (1.1) is

$$\frac{1}{\sqrt{G}} (\sqrt{G} G^{\alpha\beta} x_{,\alpha}^\mu)_{,\beta} + G^{\alpha\beta} x_{,\alpha}^\nu x_{,\beta}^\rho \Gamma_{\nu\rho}^\mu(x) = 0, \quad (1.2)$$

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where $\Gamma_{\nu\rho}^\mu$ denote the Christoffel symbols of the metric g , which vanish if \mathcal{M} is the flat Minkowski space.

Membrane equations (or relativistic strings) arise in the context of membrane, supermembrane theories and higher-dimensional extensions of string theory, where they are called p -branes according to the dimension of the spacelike object. Geometrically, membranes are time-like submanifolds with vanishing mean curvature. They are one case of an important class of geometric wave equations which is the Lorentzian analogue of the minimal submanifold equations. Since such equations possess plenty of geometric phenomenon and complicated structure, (for example, they develop singularities in finite time, and degenerate and not strictly hyperbolic properties.) much work is attracted in recent years. The case of non-compact timelike maximal graphs in Minkowski spacetimes \mathbf{R}^{1+n} is fairly well understood. Global well-posedness for sufficiently small initial data was established by Brendle [8] and by Lindblad [12]. The case of general codimension and local well-posedness in the light cone gauge were studied by Allen, Andersson and Isenberg [1] and Allen, Andersson and Restuccia [2], respectively. Kong and his collaborators [10, 11] obtained a representation formula of solution and presented many numerical evidence where singularity formation is prominent. Bellettini, Hoppe, Novaga and Orlandi [3] showed that if the initial curve is a centrally symmetric convex curve and the initial velocity is zero, the string shrinks to a point in finite time. They noticed that it should be noted that the string does not become extinct there, but rather comes out of the singularity point, evolves back to its original shape and then periodically afterwards. Most recently, Nguyen and Tian [13] showed that timelike maximal cylinders in \mathbf{R}^{1+2} always develop singularities in finite time and that, infinitesimally at a generic singularity, their time slices are evolved by a rigid motion or a self-similar motion. They also proved a mild generalization in non-flat backgrounds.

In this paper, we show that the time slice of timelike Sobolev regularity torus in Minkowski space \mathbf{R}^{1+n} always keep a rigid motion in infinite time. Let \mathbf{R}^{1+n} denote the $n + 1$ dimensional Minkowski spacetime endowed with the flat metric, and (t, x^1, \dots, x^n) its standard Cartesian coordinates. Our main result is:

Theorem 1.1. *For any immersed closed curve $\Sigma_0 \subset \mathbf{R}^{1+n} = \{t = 0\}$ and future directed timelike and nowhere vanishing vector field Ξ along Σ_0 , there exists a globally Sobolev regularity immersed surface $\Sigma \subset \mathbf{R}^{1+n}$ (the motion of the action for the surface Σ satisfies (1.1)) which is a torus \mathcal{T} and contains Σ_0 and tangential to Ξ such that its induced metric is Lorentzian and its mean curvature vector vanishes.*

Equivalently, the above result shows that for any arbitrary initial data, a closed string forms a torus, which time slice must keep a rigid motion in infinite time, i.e. they are either translated or rotated. When $n = 2$, Nguyen and Tian [13] has obtained that one evolves a closed curve in \mathbf{R}^{1+2} in a timelike direction such that its mean curvature is zero, it will form singularity in finite time. There exists the “maximal surface” after singularity forms. The singularity may be the reason why we can only obtain the Sobolev regularity torus in \mathbf{R}^{1+2} , but not smooth torus. From the point view of the equation, the “small divisor” problem is the critical point of the existence of non smooth surface.

Due to the surface being timelike, i.e. $\Lambda := \langle x_t, x_\theta \rangle^2 - (|x_t|^2 - 1)|x_\theta|^2 > 0$, direct computation shows the following Euler-Lagrange equation

$$\partial_t \left(\frac{|x_\theta|^2 x_t - \langle x_t, x_\theta \rangle x_\theta}{\sqrt{\Lambda}} \right) - \partial_\theta \left(\frac{\langle x_t, x_\theta \rangle x_t - (|x_t|^2 - 1)x_\theta}{\sqrt{\Lambda}} \right) = 0, \quad (1.3)$$

which coincide with (1.2).

By computation and (1.3), it has the following n nonlinear wave equations of Born-Infeld type

$$|x_\theta|^2 x_{tt} - 2\langle x_t, x_\theta \rangle x_{t\theta} + (|x_t|^2 - 1)x_{\theta\theta} = 0, \quad (1.4)$$

where $x = (x_1, \dots, x_n)$, $x_k = x_k(t, \theta)$ is a suitable coordinate system in parameter form for $k = 1, \dots, n$ and $(t, \theta) \in \mathbf{R}^2$. We reduce our problem to find time-space pair (t, θ) as a point of \mathbf{T}^2 such that equation (1.4) possess a solution as

$$x_k(t, \theta) = t + \theta + u_k(t + \theta, \omega t), \quad k = 1, \dots, n, \quad (1.5)$$

where \mathbf{T}^2 is a two dimensional torus, $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$, ω denotes the frequency and satisfies non-resonant condition

$$|n\omega^2 l^2 - j^2| \geq \frac{\gamma}{|l|^\tau}, \quad (l, j) \in \mathbf{Z}^2 / \{(0, 0)\}, \quad (1.6)$$

with $0 < \gamma < 1$ and $\tau > 0$. Assumption (1.6) means that the forcing frequency ω does not enter in resonance with the normal mode frequencies $\omega_j := |j|$ of oscillations of the membrane. By standard arguments, (1.6) is satisfied for all $\omega \in [\omega_1, \omega_2]$ but a subset of measure $O(\gamma)$, for fixed $0 < \omega_1 < \omega_2$.

Inserting (1.5) into (1.4), thus the motion for the displacement is governed by a nonlinear wave system

$$n\omega^2 u_{tt} - u_{yy} + \omega^2 (2 \sum_{k=1}^n u_{ky} + |u_y|^2) u_{tt} - 2\omega^2 (\sum_{k=1}^n u_{kt} + \langle u_t, u_y \rangle) u_{ty} + \omega^2 |u_t|^2 u_{yy} = 0, \quad (1.7)$$

where $u = (u_1, \dots, u_n)$, $(t, y) \in \mathbf{T}^2$ and $k = 1, \dots, n$.

Let us denote by $\mathbf{H}_s := H_s \times \dots \times H_s$ the real Sobolev space with the norm $\|U\|_s = \sum_{k=1}^n \|u_k\|_s$, $\forall U = (u_1, \dots, u_n) \in \mathbf{H}_s$, where

$$\begin{aligned} H_s &:= H_s(\mathbf{T}^2; \mathbf{R}) \\ &:= \{u(t, x) = \sum_{(l, j) \in \mathbf{Z}^2} u_{l, j} e^{i(lt + jy)}, \quad u_{l, j}^* = u_{-l, -j} \mid \|u\|_s^2 := \sum_{(l, j) \in \mathbf{Z}^2} e^{2(|l| + |j|)s} |u_{l, j}|^2 < +\infty\}, \end{aligned}$$

and $s > 1$. \mathbf{H}_s and H_s are Banach algebra with respect to the multiplication of functions, namely

$$u_1, u_2 \in \mathbf{H}_s \implies u_1 u_2 \in \mathbf{H}_s \quad \text{and} \quad \|u_1 u_2\|_s \leq \|u_1\|_s \|u_2\|_s.$$

In fact, we reduce our problem into finding time-quasi-periodic solution like (1.5) of n nonlinear wave equations of Born-Infeld type (1.4). This time-quasi-periodic solution forms a torus \mathcal{T} . The problem of finding periodic solutions for nonlinear PDEs has attracted much attention which dates back to the work of Rabinowitz [14]. He studied the existence of time-periodic solutions with a rational frequency for a one-dimensional nonlinear wave equation by variational approach. But for other frequencies of time-periodic solution, variational approach

seems weaker, due to the small divisor appears. To overcome this difficulty, one of powerful method of solving nonlinear wave equations is the analytic Newton iteration scheme combining the Lyapunov-Schmidt decomposition, which is first introduced by Craig and Wayne [9]. The main difficulty of this strategy lies in the Greens function analysis and the control of the inverse of infinite matrices with small eigenvalues. Bourgain [6, 7] extended the Craig-Wayne method to obtain the existence of quasi-periodic solutions of nonlinear wave equations and Schrödinger equations. Berti and Bolle [4] generalized previous results of Bourgain to more general nonlinearities of class of \mathbf{C}^k and assuming weaker non-resonance conditions. We notice that above results deal with nonlinear PDE with Hamiltonian structure, but nonlinear wave system (1.7) has not Hamiltonian structure. Thus a new Nash-Moser iteration scheme is need to be constructed.

To prove theorem 1.1, we first need to carry out the Lyapunov-Schmidt reduction. Instead of looking for solutions of (1.7) in a shrinking neighborhood of zero, it is a convenient device to perform the rescaling

$$u \longrightarrow \epsilon u, \quad \epsilon > 0,$$

having

$$n\omega^2 u_{tt} - u_{yy} + \epsilon\omega^2(2 \sum_{k=1}^n u_{ky} + \epsilon|u_y|^2)u_{tt} - 2\epsilon\omega^2(\sum_{k=1}^n u_{kt} + \epsilon\langle u_t, u_y \rangle)u_{ty} + \epsilon^2\omega^2|u_t|^2 u_{yy} = 0 \quad (1.8)$$

Due to any solution $u_i = \sum_{j \geq 1} v_{ij} \cos(\frac{jt}{\sqrt{n}} + \theta_j) \sin(jy)$ of the linearized equation at $u_i = 0$, $i = 1, \dots, n$,

$$n\omega^2 u_{tt} - u_{yy} = 0$$

is 2π periodic in time, equation (1.8) is called a completely resonant wave equation. In order to solve (1.8), we perform the Lyapunov-Schmidt reduction via the orthogonal decomposition

$$\mathbf{H}_s = (\mathbf{V} \cap \mathbf{H}_s) \oplus (\mathbf{W} \cap \mathbf{H}_s),$$

where \mathbf{V} is the space of the solutions of the linear equation $nu_{tt} - u_{yy} = 0$ and even in time

$$\mathbf{V} := V \times \underbrace{\dots \times V}_n, \quad V := \{v = \sum_{l \geq 1} 2 \cos(\frac{lt}{\sqrt{n}}) \sin(l y) v_l | v_l \in \mathbf{R}, \sum_{l \geq 1} |l|^2 |v_l|^2 < +\infty\},$$

\mathbf{W} is the Sobolev functions with zero mean value

$$\mathbf{W} := W \times \underbrace{\dots \times W}_n, \quad W := \{w = \sum_{(l,j) \in \mathbf{Z}^2} e^{i(lt+jy)} w_{l,j} | \langle w \rangle := \int_{\mathbf{T}^2} w(t,y) \frac{dtdy}{(2\pi)^2} = 0\}.$$

Writing $u = v + w$ with $v \in \mathbf{V}$ and $w \in \mathbf{W}$, equation (1.8) leads to deal with the following bifurcation equation

$$-\frac{(n\omega^2 - 1)}{2} \Delta v + \epsilon \Pi_{\mathbf{V}}(\omega^2(2 \sum_{k=1}^n (v_{ky} + w_{ky}) + \epsilon|v_y + w_y|^2)(v_{tt} + w_{tt}))$$

$$-2\omega^2\left(\sum_{k=1}^n(v_{kt} + w_{kt}) + \epsilon\langle v_t + w_t, v_y + w_y\rangle\right)(v_{ty} + w_{ty}) + \epsilon\omega^2|v_t + w_t|^2(v_{yy} + w_{yy})) = 0 \quad (1.9)$$

and the range equation

$$\begin{aligned} & J_\omega w + \epsilon\Pi_{\mathbf{W}}(\omega^2(2\sum_{k=1}^n(v_{ky} + w_{ky}) + \epsilon|v_y + w_y|^2)(v_{tt} + w_{tt}) \\ & -2\omega^2\left(\sum_{k=1}^n(v_{kt} + w_{kt}) + \epsilon\langle v_t + w_t, v_y + w_y\rangle\right)(v_{ty} + w_{ty}) + \epsilon\omega^2|v_t + w_t|^2(v_{yy} + w_{yy})) = 0, \end{aligned} \quad (1.10)$$

where $\Delta v = v_{tt} + v_{xx}$, $J_\omega w = n\omega^2 w_{tt} - w_{yy}$, $\Pi_{\mathbf{V}} : \mathbf{H}_s \rightarrow \mathbf{V}$ and $\Pi_{\mathbf{W}} : \mathbf{H}_s \rightarrow \mathbf{W}$ denote the projectors, respectively, on \mathbf{V} and \mathbf{W} .

To prove the existence of solution for (1.9), we need to set the frequency-amplitude relation

$$\frac{n\omega^2 - 1}{2} = 2\omega^2\epsilon,$$

then by (1.9), we have

$$\begin{aligned} & -\Delta v + \Pi_{\mathbf{V}}\left(\left(\sum_{k=1}^n(v_{ky} + w_{ky}) + \frac{\epsilon}{2}|v_y + w_y|^2\right)(v_{tt} + w_{tt})\right. \\ & \left.-\left(\sum_{k=1}^n(v_{kt} + w_{kt}) + \epsilon\langle v_t + w_t, v_y + w_y\rangle\right)(v_{ty} + w_{ty}) + \frac{\epsilon}{2}|v_t + w_t|^2(v_{yy} + w_{yy}))\right) = 0, \end{aligned} \quad (1.11)$$

Next we carry out the rescaling

$$v \longrightarrow \epsilon v, \quad \epsilon > 0,$$

we obtain

$$\begin{aligned} & -\Delta v + \epsilon\Pi_{\mathbf{V}}\left(\left(\sum_{k=1}^n(v_{ky} + w_{ky}) + \frac{\epsilon^2}{2}|v_y + w_y|^2\right)(v_{tt} + w_{tt})\right. \\ & \left.-\left(\sum_{k=1}^n(v_{kt} + w_{kt}) + \epsilon^2\langle v_t + w_t, v_y + w_y\rangle\right)(v_{ty} + w_{ty}) + \frac{\epsilon^2}{2}|v_t + w_t|^2(v_{yy} + w_{yy}))\right) = 0. \end{aligned} \quad (1.12)$$

Since the bifurcation equation (1.9) is elliptic equation with a higher derivative perturbation, we can not solve it by the contraction mapping theorem and classical implicit function theorem (or the variational method) via truncating this infinite dimensional system. To overcome the higher order derivative perturbation in bifurcation equation and the “small divisor” phenomenon in range equation, we need to construct a suitable Nash-Moser iteration scheme.

The structure of the paper is as follows: In next section, we give a crucial analysis of the linearized operators, which plays a crucial role in the Nash-Moser iteration. The last section is devoted to construct a new Nash-Moser iteration scheme to solve the range equation (1.10) and the bifurcation equation (1.12).

2 Analysis of the Linearized operator

This section will carry out analysis of linearized operator. We consider the orthogonal splitting

$$\mathbf{H}_s := \mathbf{W}^{(N_p)} \oplus \mathbf{W}^{(N_p)\perp},$$

where

$$\mathbf{W}^{(N_p)} = \{w = (w_1, \dots, w_n) \in \mathbf{H}_s | w_k = \sum_{|(l,j)| \leq N_p} w_{k,l,j} e^{i(lt+jy)}, \quad k = 1, \dots, n\}$$

and

$$\mathbf{W}^{(N_p)\perp} = \{w = (w_1, \dots, w_n) \in \mathbf{H}_s | w_k = \sum_{|(l,j)| \geq N_p} w_{k,l,j} e^{i(lt+jy)}, \quad k = 1, \dots, n\},$$

where p denotes the “ p ”th iterative step. For a given suitable $N_0 > 1$, we take $N_p \leq N_{p+1}$ and $N_p = N_0^p, \forall p \in \mathbf{N}$.

Define the projectors $\Psi^{(N_p)} : X_s \longrightarrow \mathbf{W}^{(N_p)}$. For $N > 0$, it satisfies the “smoothing” properties:

$$\begin{aligned} \|\Psi^{(N)} w\|_{s+d} &\leq N^d \|w\|_s, \quad \forall w \in \mathbf{H}_s, \quad \forall s, \quad d \geq 0, \\ \|(I - \Psi^{(N)})w\|_s &\leq N^{-d} \|w\|_{s+d}, \quad \forall w \in \mathbf{H}_{s+d}, \quad \forall s, \quad d \geq 0. \end{aligned} \quad (2.1)$$

Consider the truncation equation of (1.10)

$$\mathcal{J}(w) := J_\omega w + 2\omega^2 \epsilon \Psi^{(N_p)} f(v_t, w_t, v_y, w_y, v_{tt}, w_{tt}, v_{yy}, w_{yy}) = 0, \quad (2.2)$$

where $J_\omega w = n\omega^2 w_{tt} - w_{yy}$ and

$$\begin{aligned} f(v_t, w_t, v_y, w_y, v_{tt}, w_{tt}, v_{yy}, w_{yy}) &= \Pi_{\mathbf{W}} \left(\left(\sum_{k=1}^n (v_{ky} + w_{ky}) + \frac{\epsilon}{2} |v_y + w_y|^2 \right) (v_{tt} + w_{tt}) \right. \\ &\quad \left. - \left(\sum_{k=1}^n (v_{kt} + w_{kt}) + \epsilon \langle v_t + w_t, v_y + w_y \rangle \right) (v_{ty} + w_{ty}) + \frac{\epsilon}{2} |v_t + w_t|^2 (v_{yy} + w_{yy}) \right). \end{aligned} \quad (2.3)$$

By direct computation, the linearized operator of (2.2) has the following form

$$\mathcal{J}_\omega^{(N_p)} := \Psi^{(N_p)} (\mathcal{J}_\omega + 2\omega^2 \epsilon D_w f(v_t, w_t, v_y, w_y, v_{tt}, w_{tt}, v_{yy}, w_{yy}))|_{\mathbf{W}^{(N_p)}}, \quad (2.4)$$

where

$$\begin{aligned} &D_w f(v_t, w_t, v_y, w_y, v_{tt}, w_{tt}, v_{yy}, w_{yy}) h \\ &= \left(\sum_{k=1}^n (v_{ky} + w_{ky}) + \frac{\epsilon}{2} |v_y + w_y|^2 \right) h_{tt} - \left(\sum_{k=1}^n (v_{ky} + w_{ky}) + \epsilon \langle v_t + w_t, v_y + w_y \rangle \right) h_{ty} \\ &\quad + \frac{\epsilon}{2} |v_t + w_t|^2 h_{yy} + \sum_{k=1}^n h_{ky} (v_{tt} + w_{tt}) + 2\epsilon \langle v_y + w_y, h_y \rangle (v_{tt} + w_{tt}) \\ &\quad - \sum_{k=1}^n h_{kt} (v_{ty} + w_{ty}) - \epsilon (\langle v_t + w_t, h_y \rangle + \langle h_t, v_y + w_y \rangle) (v_{ty} + w_{ty}) \\ &\quad + \epsilon \langle v_t + w_t, h_t \rangle (v_{yy} + w_{yy}). \end{aligned} \quad (2.5)$$

Lemma 2.1. *For any $s > 0$, there exist constants $C_{1\epsilon}$, $C_{2\epsilon}$, C_3 and $C_{4\epsilon}$ such that*

$$\|\Psi^{(N_p)} D_w f(v_t, w_t, v_y, w_y, v_{tt}, w_{tt}, v_{yy}, w_{yy})h\|_s \leq C_\epsilon N_p^4 ((\|w\|_s^2 + \|v\|_s^2)\|h\|_s + 2(\|v\|_s + \|w\|_s)\|h\|_s), \quad (2.6)$$

$$\begin{aligned} & \|\Psi^{(N_p)}(f(v_t, w_t + h_t, v_y, w_y + h_y, v_{tt}, w_{tt} + h_{tt}, v_{yy}, w_{yy} + h_{yy}) \\ & \quad - f(v_t, w_t, v_y, w_y, v_{tt}, w_{tt}, v_{yy}, w_{yy}) - D_w f(v_t, w_t, v_y, w_y, v_{tt}, w_{tt}, v_{yy}, w_{yy})h)\|_s \\ & \leq C_{2\epsilon} N_p^4 (C_3 + \|v\|_s + \|w\|_s)(\|h\|_s^2 + \|h\|_s^3), \end{aligned} \quad (2.7)$$

$$\|\Psi^{(N_p)} f(v_t, w_t, v_y, w_y, v_{tt}, w_{tt}, v_{yy}, w_{yy})\|_s \leq C_{4\epsilon} N_p^4 (\|v\|_s^3 + \|w\|_s^3). \quad (2.8)$$

Proof. By direct computation, we have

$$\begin{aligned} & \|\Psi^{(N_p)} \sum_{k=1}^n (v_{ky} + w_{ky})h_{tt}\|_s = \sum_{k=1}^n \|\Psi^{(N_p)} \sum_{k=1}^n (v_{ky} + w_{ky})h_{ktt}\|_s \\ & = \sum_{k=1}^n \left\| - \left(\sum_{k=1}^n \sum_{|(l,j)| \leq N_p} j(iv_{k,l,j} + iw_{k,l,j})e^{i(lt+jx)} \right) \left(\sum_{|(l,j)| \leq N_p} l^2 h_{k,l,j} e^{i(lt+jx)} \right) \right\|_s \\ & \leq \sum_{k=1}^n \left\| - \sum_{k=1}^n \sum_{|(l,j)| \leq N_p} j(iv_{k,l,j} + iw_{k,l,j})e^{i(lt+jx)} \right\|_s \left\| \sum_{|(l,j)| \leq N_p} l^2 h_{k,l,j} e^{i(lt+jx)} \right\|_s \\ & \leq N_p^3 \sum_{k=1}^n \left(\sum_{k=1}^n \sum_{(l,j) \in \mathbf{Z}^2} (|v_{k,l,j}| + |w_{k,l,j}|) e^{(s)(|l|+|j|)} \right) \left(\sum_{(l,j) \in \mathbf{Z}^2} |h_{k,l,j}| e^{(s)(|l|+|j|)} \right) \\ & \leq N_p^3 \sum_{k=1}^n (\|v\|_s + \|w\|_s) \|h_k\|_s^2 = N_p^3 (\|v\|_s + \|w\|_s) \|h\|_s \end{aligned} \quad (2.9)$$

and

$$\begin{aligned} \|\Psi^{(N_p)} |v_y + w_y|^2 h_{tt}\|_s &= \sum_{k=1}^n \|\Psi^{(N_p)} |v_y + w_y|^2 h_{ktt}\|_s \leq \sum_{k=1}^n \|\Psi^{(N_p)} |v_y + w_y|^2\|_s \|\Psi^{(N_p)} h_{ktt}\|_s \\ &\leq \sum_{k=1}^n (\Psi^{(N_p)} \|\langle v_y, v_y \rangle\|_s + 2\|\Psi^{(N_p)} \langle v_y, w_y \rangle\|_s + \|\Psi^{(N_p)} \langle w_y, w_y \rangle\|_s) \|\Psi^{(N_p)} h_{ktt}\|_s \\ &\leq N_p^4 \sum_{k=1}^n (\|v\|_s^2 + 2\|v\|_s \|w\|_s + \|w\|_s^2) \|h_k\|_s \\ &\leq N_p^4 (\|v\|_s^2 + 2\|v\|_s \|w\|_s + \|w\|_s^2) \|h\|_s. \end{aligned} \quad (2.10)$$

Using the similar computation method, we obtain

$$\begin{aligned} & \|\Psi^{(N_p)} \sum_{k=1}^n (v_{ky} + w_{ky})h_{ty}\|_s \leq N_p^3 (\|v\|_s + \|w\|_s) \|h_k\|_s, \\ & \|\Psi^{(N_p)} \langle v_t + w_t, v_y + w_y \rangle h_{ty}\|_s, \| |v_t + w_t|^2 h_{yy} \|_s \leq N_p^4 (\|v\|_s^2 + 2\|v\|_s \|w\|_s + \|w\|_s^2) \|h\|_s, \end{aligned}$$

$$\begin{aligned}
& \|\Psi^{(N_p)} \sum_{k=1}^n h_{ky}(v_{tt} + w_{tt})\|_s, \quad \|\Psi^{(N_p)} \sum_{k=1}^n h_{kt}(v_{ty} + w_{ty})\|_s \leq N_p^3 \|h\|_s (\|v\|_s + \|w\|_s), \\
& \|\Psi^{(N_p)} \langle v_y + w_y, h_y \rangle (v_{tt} + w_{tt})\|_s, \quad \|\Psi^{(N_p)} \langle v_t + w_t, h_y \rangle (v_{ty} + w_{ty})\|_s \leq N_p^4 (\|v\|_s + \|w\|_s)^2 \|h\|_s, \\
& \|\Psi^{(N_p)} \langle h_t, v_y + w_y \rangle (v_{ty} + w_{ty})\|_s, \quad \|\Psi^{(N_p)} \langle v_t + w_t, h_t \rangle (v_{yy} + w_{yy})\|_s \leq N_p^4 (\|v\|_s + \|w\|_s)^2 \|h\|_s.
\end{aligned}$$

Above estimate combining with the Young inequality give that

$$\|\Psi^{(N_p)} D_w f(v_t, w_t, v_y, w_y, v_{tt}, w_{tt}, v_{yy}, w_{yy}) h\|_s \leq C_{1\epsilon} N_p^4 ((\|w\|_s^2 + \|v\|_s^2) \|h\|_s + 2(\|v\|_s + \|w\|_s) \|h\|_s),$$

where $C_{1\epsilon}$ denotes a constant depending on ϵ .

Next we prove (2.7) and (2.8). By (2.3) and (2.5), we derive

$$\begin{aligned}
& f(v_t, w_t + h_t, v_y, w_y + h_y, v_{tt}, w_{tt} + h_{tt}, v_{yy}, w_{yy} + h_{yy}) - f(v_t, w_t, v_y, w_y, v_{tt}, w_{tt}, v_{yy}, w_{yy}) \\
& \quad - D_w f(v_t, w_t, v_y, w_y, v_{tt}, w_{tt}, v_{yy}, w_{yy}) h \\
& = \left(\sum_{k=1}^n h_{ky} \right) h_{tt} + \frac{\epsilon}{2} (2 \langle v_y + w_y, h_y \rangle + |h_y|^2) h_{tt} + \epsilon |h_y|^2 (v_{tt} + w_{tt}) \\
& \quad + \left(\sum_{k=1}^n h_{kt} \right) h_{ty} + \epsilon ((\langle v_t + w_t, h_y \rangle + \langle v_y + w_y, h_t \rangle) h_{ty} + \langle h_t, h_y \rangle (v_{ty} + w_{ty})) \\
& \quad + 2\epsilon (\langle v_t + w_t, h_t \rangle + |h_t|^2) h_{yy}.
\end{aligned}$$

By the similar estimate with (2.9)-(2.10), we obtain

$$\begin{aligned}
& \|\Psi^{(N_p)} (f(v_t, w_t + h_t, v_y, w_y + h_y, v_{tt}, w_{tt} + h_{tt}, v_{yy}, w_{yy} + h_{yy}) \\
& \quad - f(v_t, w_t, v_y, w_y, v_{tt}, w_{tt}, v_{yy}, w_{yy}) - D_w f(v_t, w_t, v_y, w_y, v_{tt}, w_{tt}, v_{yy}, w_{yy}) h) \|_s \\
& \leq C_{2\epsilon} N_p^4 (C_3 + \|v\|_s + \|w\|_s) (\|h\|_s^2 + \|h\|_s^3),
\end{aligned}$$

where $C_{2\epsilon}$ and C_3 are constants, $C_{2\epsilon}$ depends on ϵ . The estimate (2.8) is similar, so we omit it. This completes the proof. \square

In the following, we study the property of operators $\mathcal{J}_\omega^{(N_p)}$. To obtain the same separation properties as in [4, 6], we shall assume that

$$|\omega^2 q - p| \geq \frac{\gamma}{\max\{1, |p|^\mu\}}, \quad \forall (q, p) \in \mathbf{Z}^2 / \{(0, 0)\}, \quad \forall \mu > 1. \quad (2.11)$$

Lemma 2.2. *Assume that ω satisfies (1.6) and (2.11). For $s_2 > s_1 \geq 0$, the linearized operator $\mathcal{J}_\omega^{(N_p)}$ satisfies*

$$\begin{aligned}
& \|(\mathcal{J}_\omega^{(N_p)}(\epsilon, w))^{-1} h\|_{s_1} \\
& \leq C(s_2 - s_1) N_p^{\tau + \kappa_0} (1 + \epsilon \varsigma^{-1} C_{\epsilon, s_1} N_p^4 (\|w\|_{s_2 + s'}^2 + \|v\|_{s_2 + s'}^2 + 2\|w\|_{s_2 + s'} + 2\|v\|_{s_2 + s'}))^3 \|h\|_{s_2}, \quad (2.12)
\end{aligned}$$

where $C(s_2 - s_1) = c(s_2 - s_1)^{-\tau}$, $c = c(\varsigma, \tau, s_1, s_2, \gamma_1, \gamma)$ and C_ϵ denote constant, s' is a constant satisfying $s' \geq \frac{7}{2}$.

In what follows, we carry out proving Lemma 2.2. Let

$$b(t, x) := (\partial_w f)(v_t(t, x), w_t(t, x), v_y(t, x), w_y(t, x), v_{tt}(t, x), w_{tt}(t, x), v_{yy}(t, x), w_{yy}(t, x)).$$

For notational convenience, we denote $N = N_p$. For fixing $\varsigma > 0$, we define the regular sites R and the singular sites S as

$$R := \{a \in \Omega_N \mid |d_{(l,j)}| \geq \varsigma\} \text{ and } S := \{a \in \Omega_N \mid |d_{(l,j)}| < \varsigma\}. \quad (2.13)$$

Due to the orthogonal decomposition $\mathbf{H}^{(N)} = \mathbf{H}_R \oplus \mathbf{H}_S$, we identify a linear operator A acting on \mathbf{H}_s with its matrix representation $A = (A_b^a)_{a,b \in \Omega_N}$ with blocks $A_b^a \in \mathcal{J}(\mathcal{N}_a, \mathcal{N}_b)$. We define the polynomially localized block matrices

$$\mathcal{A}_s := \{A = (A_b^a)_{a,b \in \Omega_N} : |A|_s^2 := \sup_{b \in \Omega_N} \sum_{a \in \Omega_N} e^{2s|b-a|} \|A_b^a\|_0^2 < \infty\},$$

where $\|A_b^a\|_0 := \sup_{w \in \mathcal{N}_a, \|w\|_0=1} \|A_b^a w\|_0$ is the \mathbf{L}^2 -operator norm in $\mathcal{J}(\mathcal{N}_a, \mathcal{N}_b)$, for $\mathcal{N}_a, \mathcal{N}_b \subset \Omega_N$. If $s' > s$, then these holds $\mathcal{A}_{s'} \subset \mathcal{A}_s$.

The next lemma (see [5]) shows the algebra property of \mathcal{A}_s and interpolation inequality.

Lemma 2.3. *There holds*

$$|AB|_s \leq c(s)|A|_s|B|_s, \quad \forall A, B \in \mathcal{A}_s, \quad s > s_0 > \frac{3}{2}, \quad (2.14)$$

$$|AB|_s \leq c(s)(|A|_s|B|_{s_0} + |A|_{s_0}|B|_s), \quad s \geq s_0, \quad (2.15)$$

$$\|Au\|_s \leq c(s)(|A|_s\|u\|_{s_0} + |A|_{s_0}\|u\|_s), \quad \forall u \in \mathbf{H}_s, \quad s \geq s_0. \quad (2.16)$$

By Lemma 2.3, we can get, $\forall m \in \mathbf{N}$,

$$|A^m|_s \leq c(s)^{m-1}|A|_s^m, \quad (2.17)$$

$$|A^m|_s \leq m(c(s)|A|_{s_0})^{m-1}|A|_s. \quad (2.18)$$

The next two lemmas can be obtained by a small modification of the proof of Lemma 2.3.9 in [4] and Proposition 2.19 in [5], so we omit it.

Lemma 2.4. *Let $A \in \mathcal{A}_s$, $\Omega_1, \Omega_2 \subset \Omega_N$, and $\Omega_1 \cap \Omega_2 = \emptyset$. Then*

$$\|A_{\Omega_2}^{\Omega_1}\|_0 \leq c(s)|A|_s d^{-1}(\Omega_1, \Omega_2)^{2s-3}.$$

Since \mathbf{H}_s is an algebra, for each $b \in \mathbf{H}_s$ defines the multiplication operator

$$w(t, x) \mapsto b(t, x)w(t, x), \quad \forall w \in \mathbf{H}_s, \quad (2.19)$$

which is represented by $(B_b^a)_{a,b \in \Omega_N}$ with $B_b^a := \Psi_{\mathcal{N}_b} b(t, x)|_{\mathcal{N}_a} \in \mathcal{J}(\mathcal{N}_a, \mathcal{N}_b)$.

Lemma 2.5. *For real functions $b(t, x) \in \mathbf{H}_{s+s'}$ with $s' \geq \frac{7}{2}$, the matrix $(B_b^a)_{a,b \in \Omega_N}$ representing the multiplication operator (2.19) is self-adjoint, it belongs to the algebra of polynomially localized matrices \mathcal{A}_s , and we have*

$$|B|_s \leq K(s)\|b\|_{s+s'},$$

where $K(s)$ is a constant depending on s .

We define

$$h \mapsto \mathcal{J}^{(N)}[h] := \Psi^{(N)}(\mathcal{J}_\omega h + 2\omega^2 \epsilon b(t, x)h), \quad \forall h \in \mathbf{H}^{(N)}. \quad (2.20)$$

We write (2.20) by the block matrix

$$\mathcal{J}_\omega^{(N)} = D + 2\omega^2 \epsilon T, \quad D := \text{diag}_{a=(l,j) \leq \Omega_N} (d_{(l,j)}), \quad (2.21)$$

where $(l, j) \in \mathbf{Z}^2$, $\Omega_N := \{a := (l, j) \in \mathbf{Z}^2 \mid |(l, j)| \leq N\}$,

$$d_{(l,j)} := n\omega^2 l^2 - j^2, \quad (2.22)$$

and for $\mathcal{N}_a, \mathcal{N}_b \subset \Omega_N$,

$$T := (T_b^a)_{a,b \in \Omega_N}, \quad T_b^a := \Psi_{\mathcal{N}_b} \mathcal{J}_\omega^N|_{\mathcal{N}_a} \in \mathcal{J}(\mathcal{N}_a, \mathcal{N}_b). \quad (2.23)$$

Here T is represented by the self-adjoint Toeplitz matrix $(T_{a-b})_{a,b \in \Omega_N}$, the T_a being the Fourier coefficients of the function $b(t, x)$.

In what follows, we prove the estimate (2.12). For each N , we denote the restrictions of S, R, Ω_α to Ω_N with the same symbols. The following result shows the separation of singular sites, and the proof can be completed by following essentially the scheme of [4, 5, 6], so we omit it.

Lemma 2.6. *Assume that ω satisfies (1.6) and (2.11). There exists $\varsigma_0(\gamma)$ such that for $\varsigma \in (0, \varsigma_0(\gamma))$ and a partition of the singular sites S which can be partitioned in pairwise disjoint clusters Ω_α as*

$$S = \bigcup_{\alpha \in A} \Omega_\alpha \quad (2.24)$$

satisfying

- (dyadic) $\forall \alpha \in A \subset \Omega_N$, $M_\alpha \leq 2m_\alpha$, where $M_\alpha := \max_{a \in \Omega_\alpha} |a|$, $m_\alpha := \max_{a \in \Omega_\alpha} |a|$.
- (separation) $\exists \lambda, c > 0$ such that $d(\Omega_\alpha, \Omega_\beta) \geq c(M_\alpha + M_\beta)^\lambda$, $\forall \alpha \neq \beta$, where $d(\Omega_\alpha, \Omega_\beta) := \max_{a \in \Omega_\alpha, b \in \Omega_\beta} |a - b|$ and $\lambda \in (0, 1)$.

Using Lemma 2.5 and Lemma 2.1, we have the following.

Lemma 2.7. *Let $s' \geq \frac{7}{2}$. For a real $b(t, x) \in \mathbf{H}_{s+s'}$, the matrix $T = (T_b^a)_{a,b \in \Omega_N}$ defined in (2.23) is self-adjoint and belongs to the algebra of polynomially localized matrices \mathcal{A}_s with*

$$|T|_s \leq K(s) \|b\|_{s+s'} \leq C_\epsilon K(s) N^4 \left(\|w\|_{s+s'}^2 + \|v\|_{s+s'}^2 + 2\|w\|_{s+s'} + 2\|v\|_{s+s'} \right).$$

Moreover, for any $s > s'$,

$$|T|_s \leq K'(s) N^{s'} \|b\|_s \leq C_\epsilon K'(s) N^{4+s'} \left(\|w\|_{s+s'}^2 + \|v\|_{s+s'}^2 + 2\|w\|_{s+s'} + 2\|v\|_{s+s'} \right),$$

where $K(s)$ is a constant depending on s .

Since the decomposition

$$\mathbf{H}^{(N)} := \mathbf{H}_R \oplus \mathbf{H}_S,$$

with

$$\mathbf{H}_R := \bigoplus_{\alpha \in R \cap \Omega_N} \mathcal{N}_\alpha, \quad \mathbf{H}_S := \bigoplus_{\alpha \in S \cap \Omega_N} \mathcal{N}_\alpha,$$

we can represent the operator $\mathcal{J}_\omega^{(N)}$ as the self-adjoint block matrix

$$\mathcal{J}_\omega^{(N)} = \begin{pmatrix} J_R & J_R^S \\ J_S^R & J_S \end{pmatrix},$$

where $J_R^S = (J_S^R)^\dagger$, $J_R = J_R^\dagger$, $J_S = J_S^\dagger$.

Thus the invertibility of $\mathcal{J}_\omega^{(N)}$ can be expressed via the "resolvent-type" identity

$$(\mathcal{J}_\omega^{(N)})^{-1} = \begin{pmatrix} I & -J_R^{-1}J_R^S \\ 0 & I \end{pmatrix} \begin{pmatrix} J_R^{-1} & 0 \\ 0 & \mathcal{J}^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -J_S^R J_R^{-1} & I \end{pmatrix}, \quad (2.25)$$

where the "quasi-singular" matrix

$$\mathcal{J} := J_S - J_S^R J_R^{-1} J_R^S \in \mathcal{A}_s(S),$$

where $\mathcal{A}_s(S)$ denotes \mathcal{A}_s restricting on S . The reason of $\mathcal{J} \in \mathcal{A}_s(S)$ is that \mathcal{J} is the restriction to S of the polynomially localized matrix

$$I_S(J - I_S J I_R \tilde{J}^{-1} I_R J I_S) I_S \in \mathcal{A}_s,$$

where

$$\tilde{J}^{-1} = \begin{pmatrix} I & 0 \\ 0 & J_R \end{pmatrix}.$$

Lemma 2.8. *Assume that ω satisfies (1.6) and (2.11). For $s_0 < s_1 < s_2$, $|J_R^{-1}|_{s_0} \leq 2\varsigma^{-1}$, the operator J_R satisfies*

$$|\tilde{J}_R^{-1}|_{s_1} \leq c(s_1)(1 + \varepsilon\varsigma^{-1}|T|_{s_1}), \quad (2.26)$$

$$\|J_R^{-1}w\|_{s_1} \leq c(\gamma, \tau, s_2)(s_2 - s_1)^{-\tau}(1 + \varepsilon\varsigma^{-1}|T|_{s_2})\|w\|_{s_2}, \quad (2.27)$$

where $\tilde{J}^{-1} = J_R^{-1}D_R$, $c(\gamma, \tau, s_2)$ is a constant depending on γ, τ, s_2 .

Proof. It follows from (2.21) and (2.13) that D_R is a diagonal matrix and satisfies $|D_R^{-1}|_s \leq \varsigma^{-1}$. By (2.14), we have that the Neumann series

$$\tilde{J}_R^{-1} = J_R^{-1}D_R = \sum_{m \geq 0} (-\varepsilon)^m (D_R^{-1}T_R)^m \quad (2.28)$$

is totally convergent in $|\cdot|_{s_1}$ with $|J_R^{-1}|_{s_0} \leq 2\varsigma^{-1}$, by taking $\varepsilon\varsigma^{-1}|T|_{s_0} \leq c(s_0)$ small enough.

Using (2.14) and (2.18), we have that $\forall m \in \mathbf{N}$,

$$\begin{aligned} \varepsilon^m |(D_R^{-1} T_R)^m|_{s_1} &\leq \varepsilon^m c(s) |(D_R^{-1} T_R)^m|_{s_1} \\ &\leq c(s) \varepsilon^m m (c(s) |D_R^{-1} T_R|_{s_0})^{m-1} |D_R^{-1} T_R|_{s_1} \\ &\leq c'(s) \varepsilon m \varsigma^{-1} (\varepsilon c(s_1) \varsigma^{-1} |T|_{s_0})^{m-1} |T|_{s_1}, \end{aligned}$$

which together with (2.28) implies that for $\varepsilon \varsigma^{-1} |T|_{s_0} < c(s_0)$ small enough, (2.26) holds.

By nonresonance condition (1.6) and $\sup_{x>0} (x^y e^{-x}) = (y e^{-1})^y, \forall y \geq 0$, we derive

$$\begin{aligned} e^{-2(|l|+|j|)(s_2-s_1)} |n\omega^2 l^2 - j^2|^{-2} &\leq \gamma^{-1} |l|^\tau e^{-2(|l|+|j|)(s_2-s_1)} \\ &\leq c(\gamma, \tau) (s_2 - s_1)^{-2\tau}. \end{aligned} \quad (2.29)$$

Then by (2.29), for any $w \in \mathbf{H}_R$,

$$\begin{aligned} \|J_R^{-1} w\|_{s_1}^2 &= \sum_{(l,j) \in R \cap \Omega_N} e^{2(|l|+|j|)s_1} \|J_R^{-1} w_j\|_{\mathbf{L}^2}^2 \\ &\leq \sum_{(l,j) \in R \cap \Omega_N} e^{2(|l|+|j|)s_1} |n\omega^2 l^2 - j^2|^{-2} \|\tilde{J}_R^{-1} w_j\|_{\mathbf{L}^2}^2 \\ &\leq \sum_{(l,j) \in R \cap \Omega_N} e^{-2(|l|+|j|)(s_2-s_1)} |n\omega^2 l^2 - j^2|^{-2} e^{2(|l|+|j|)s_2} \|\tilde{J}_R^{-1} w_j\|_{\mathbf{L}^2}^2 \\ &\leq c(\gamma, \tau) (s_2 - s_1)^{-2\tau} \|\tilde{J}_R^{-1} w\|_{s_2}^2. \end{aligned}$$

Thus using interpolation (2.16) and (2.26), we derive that for $s_1 < s < s_2$,

$$\begin{aligned} \|J_R^{-1}\|_{s_1} &\leq c(\gamma, \tau) (s_2 - s_1)^{-\tau} \|\tilde{J}_R^{-1} w\|_{s_2} \\ &\leq c(r, \tau, s_2) (s_2 - s_1)^\tau (|\tilde{J}_R^{-1}|_{s_2} \|w\|_s + |\tilde{J}_R^{-1}|_s \|w\|_{s_2}) \\ &\leq c(r, \tau, s_2) (s_2 - s_1)^\tau (1 + \varepsilon \varsigma^{-1} |T|_{s_2}) \|w\|_{s_2}. \end{aligned}$$

This completes the proof. \square

Next we analyse the quasi-singular matrix \mathcal{J} . By (2.24), the singular sites restricted to Ω_N are

$$S = \bigcup_{\alpha \in l_N} \Omega_\alpha, \text{ where } l_N := \{\alpha \in \mathbf{N} | m_\alpha \leq N\},$$

and $\Omega_\alpha \equiv \Omega_\alpha \cup \Omega_N$. Due to the decomposition $\tilde{H}_S := \bigoplus_{\alpha \in l_N} \tilde{H}_\alpha$, where $\mathbf{H}_\alpha := \bigoplus_{a \in \Omega_\alpha} \mathcal{N}_a$, we represent \mathcal{J} as the block matrix $\mathcal{J} = (\mathcal{J}_\alpha^\beta)_{\alpha, \beta \in l_N}$, where $\mathcal{J}_\alpha^\beta := \Psi_{\mathbf{H}_\alpha} \mathcal{J}|_{\mathbf{H}_\beta}$. So we can rewrite

$$\mathcal{J} = \mathcal{D} + \mathcal{T},$$

where $\mathcal{D} := \text{diag}_{\alpha \in l_N} (\mathcal{J}_\alpha)$, $\mathcal{J}_\alpha := \mathcal{J}_\alpha^\alpha$, $\mathcal{T} := (\mathcal{J}_\alpha^\beta)_{\alpha \neq \beta}$.

We define a diagonal matrix corresponding to the matrix \mathcal{D} as $\bar{D} := \text{diag}_{\alpha \in l_N} (\bar{J}_\alpha)$, where $\bar{J}_\alpha = \text{diag}_{j \in \Omega_\alpha} (D_j)$.

Lemma 2.9. Assume that ω satisfies (1.6) and (2.11). We have

$$\|\mathcal{D}^{-1}\bar{D}w\|_{s_1} \leq c(\varsigma, s_1, \gamma_1)N^\tau\|w\|_{s_2},$$

where $c(\varsigma, s_1, \gamma_1)$ is a constant which depends on ς , s_1 and γ_1 .

Proof. Note that $\|w_\alpha\|_0 \leq m_\alpha^{-s_1}\|w_\alpha\|_{s_1}$ and $M_\alpha = 2m_\alpha$. So for any $w = \sum_{\alpha \in l_N} w_\alpha \in \mathbf{H}_\alpha$, $w_\alpha \in \mathbf{H}_\alpha$,

$$\begin{aligned} \|\mathcal{D}^{-1}\bar{D}w\|_{s_1}^2 &= \sum_{\alpha \in l_N} \|\mathcal{J}_\alpha^{-1}\bar{J}_\alpha w_\alpha\|_{s_1}^2 \leq \sum_{\alpha \in l_N} M_\alpha^{2s_1} \|\mathcal{J}_\alpha^{-1}\bar{J}_\alpha w_\alpha\|_0^2 \\ &\leq c\gamma_1^{-2} \sum_{\alpha \in l_N} M_\alpha^{2(s_1+\tau)} \|\bar{J}_\alpha w_\alpha\|_0^2 \\ &\leq c\gamma_1^{-2} \sum_{\alpha \in l_N} M_\alpha^{2(s_1+\tau)} m_\alpha^{-2s_1} \|\bar{J}_\alpha w_\alpha\|_{s_1}^2 \\ &\leq c\gamma_1^{-2} 4^{s_1} \sum_{\alpha \in l_N} M_\alpha^{2\tau} \|\bar{J}_\alpha w_\alpha\|_{s_1}^2 \\ &\leq c\gamma_1^{-2} 4^{s_1} N^{2\tau} \sum_{\alpha \in l_N} \|\bar{J}_\alpha w_\alpha\|_{s_1}^2 \\ &= c\gamma_1^{-2} 4^{s_1} N^{2\tau} \|\bar{D}w\|_{s_1}^2. \end{aligned} \tag{2.30}$$

Using interpolation (2.16) and (2.13), for $0 < s_1 < s_2$, it follows from (2.30) that

$$\begin{aligned} \|\mathcal{D}^{-1}\bar{D}w\|_{s_1} &\leq c\gamma_1^{-1} 2^{s_1} N^\tau \|\bar{D}w\|_{s_1} \\ &\leq c\gamma_1^{-1} 2^{s_1} N^\tau (|\bar{D}|_{s_2} \|w\|_{s_1} + |\bar{D}|_{s_1} \|w\|_{s_2}) \\ &\leq c(\varsigma) \gamma_1^{-1} 2^{s_1+1} N^\tau \|w\|_{s_2}. \end{aligned}$$

This completes the proof. \square

The following result is taken from [5], so we omit the proof.

Lemma 2.10. For $\kappa_0 = \tau + 3$, $\forall s \geq 0$, $\forall m \in \mathbf{N}$, there hold:

$$c(s_1) \|\mathcal{D}^{-1}\mathcal{T}\|_{s_0} < \frac{1}{2}, \quad \|\mathcal{D}^{-1}\|_s \leq c(s) \gamma_1^{-1} N^\tau, \tag{2.31}$$

$$\|(\mathcal{D}^{-1}\mathcal{T})^m w\|_s \leq (\varepsilon \gamma^{-1} K(s))^m (m N^{\kappa_0} |T|_s |T|_{s_0}^{m-1} \|w\|_{s_0} + |T|_{s_0}^m \|w\|_s). \tag{2.32}$$

Lemma 2.11. Assume that ω satisfies (1.6) and (2.11). For $0 < s_0 < s_1 < s_2 < s_3$, we have

$$\|\mathcal{J}^{-1}w\|_{s_1} \leq c(\varsigma, \tau, s_1, \gamma_1, \gamma) N^{\tau+\kappa_0} (s_3 - s_2)^{-\tau} (\|w\|_{s_3} + \varepsilon |T|_{s_1} \|w\|_{s_2}). \tag{2.33}$$

Proof. The Neumann series

$$\mathcal{J}^{-1} = (I + \mathcal{D}^{-1}\mathcal{T})^{-1} \mathcal{D}^{-1} = \sum_{m \geq 0} (-1)^m (\mathcal{D}^{-1}\mathcal{T})^m \mathcal{D}^{-1} \tag{2.34}$$

is totally convergent in operator norm $\|\cdot\|_{s_0}$ with $\|\mathcal{J}^{-1}\|_{s_0} \leq c\gamma_1^{-1} N^\tau$, by using (2.31).

By (2.32) and (2.34), we have

$$\begin{aligned}
\|\mathcal{J}^{-1}w\|_{s_1} &\leq \|\mathcal{D}^{-1}w\|_{s_1} + \sum_{m \geq 1} \|(\mathcal{D}^{-1}\mathcal{T})^m \mathcal{D}^{-1}w\|_{s_1} \\
&\leq \|\mathcal{D}^{-1}w\|_{s_1} + \|\mathcal{D}^{-1}w\|_{s_1} \sum_{m \geq 1} (\varepsilon \gamma_1^{-1} K(s) |T|_{s_0})^m \\
&\quad + N^{\kappa_0} K(s_1) \varepsilon \gamma_1^{-1} |T|_{s_1} \|\mathcal{D}^{-1}w\|_{s_0} \sum_{m \geq 1} m (K(s) \varepsilon \gamma_1^{-1} |T|_{s_0})^{m-1}. \quad (2.35)
\end{aligned}$$

Using $\sup_{x>0} (x^y e^{-x}) = (ye^{-1})^y$, $\forall y \geq 0$, for $0 < s_1 < s_2 < s_3$, it follows from Lemma 2.9 that

$$\begin{aligned}
\|\mathcal{D}^{-1}w\|_{s_1}^2 &= \|\mathcal{D}^{-1}\bar{D}\bar{D}^{-1}w\|_{s_1}^2 \leq c^2(\varsigma, s_1, \gamma_1) N^{2\tau} \|\bar{D}^{-1}w\|_{s_2}^2 \\
&= c^2(\varsigma, s_1, \gamma_1) N^{2\tau} \sum_{(l,j) \in R \cap \Omega_N} e^{2(|l|+|j|)s_2} \|\bar{D}^{-1}w_j\|_{\mathbf{L}^2}^2 \\
&\leq c^2(\varsigma, s_1, \gamma_1) N^{2\tau} \sum_{(l,j) \in R \cap \Omega_N} e^{2(|l|+|j|)s_2} |n\omega^2 l^2 - j^2|^{-2} \|w_j\|_{\mathbf{L}^2}^2 \\
&\leq c^2(\varsigma, s_1, \gamma_1) N^{2\tau} \sum_{(l,j) \in R \cap \Omega_N} e^{-2(|l|+|j|)(s_3-s_2)} |l|^{-2} e^{2(|l|+|j|)s_3} \|w_j\|_{\mathbf{L}^2}^2 \\
&\leq c^2(\varsigma, \tau, s_1, \gamma_1, \gamma) N^{2\tau} (s_3 - s_2)^{-2\tau} \|w\|_{s_3}^2. \quad (2.36)
\end{aligned}$$

Thus by (2.35) and (2.36), we derive

$$\begin{aligned}
\|\mathcal{J}^{-1}w\|_{s_1} &\leq \gamma_1^{-1} N^{\kappa_0} K'(s_1) (\|\mathcal{D}^{-1}w\|_{s_1} + \varepsilon |T|_{s_1} \|\mathcal{D}^{-1}w\|_{s_0}) \\
&\leq c(\varsigma, \tau, s_1, \gamma_1, \gamma) N^{\tau+\kappa_0} (s_3 - s_2)^{-\tau} (\|w\|_{s_3} + \varepsilon |T|_{s_1} \|w\|_{s_2}), \quad (2.37)
\end{aligned}$$

where $0 < s_1 < s_2 < s_3$ and $\varepsilon \gamma_1^{-1} \varsigma^{-1} (1 + |T|_{s_0}) \leq c(k)$ small enough. \square

Now we are ready to prove Lemma 2.2. Let $w = w_R + w_S$ with $w_S \in \mathbf{H}_S$, $w_R \in \mathbf{H}_R$. Then by the resolvent identity (2.25),

$$\begin{aligned}
\|(\mathcal{J}_\omega^{(N)})^{-1}w\|_{s_1} &\leq \|J_R^{-1}w_R + J_R^{-1}J_S^R \mathcal{J}^{-1}(w_S + J_R^S J_R^{-1}w_R)\|_{s_1} + \|\mathcal{J}^{-1}(w_R + J_R^S J_R^{-1}w_R)\|_{s_1} \\
&\leq \|J_R^{-1}w_R\|_{s_1} + \|J_R^{-1}J_S^R \mathcal{J}^{-1}w_S\|_{s_1} + \|J_R^{-1}J_S^R \mathcal{J}^{-1}J_R^S J_R^{-1}w_R\|_{s_1} \\
&\quad + \|\mathcal{J}^{-1}w_R\|_{s_1} + \|\mathcal{J}^{-1}J_R^S J_R^{-1}w_R\|_{s_1}. \quad (2.38)
\end{aligned}$$

Next we estimate the right hand side of (2.38) one by one. Using (2.16), (2.27) and (2.33), for $0 < s_1 < s_2 < s_3 < s_4$, we have

$$\begin{aligned}
\|J_R^{-1}J_S^R \mathcal{J}^{-1}w_S\|_{s_1} &\leq c(\gamma, \tau, s_2) (s_2 - s_1)^{-\tau} (1 + \varepsilon \varsigma^{-1} |T|_{s_2}) \|J_S^R \mathcal{J}^{-1}w_S\|_{s_2} \\
&\leq c(\gamma, \tau, s_2) (s_2 - s_1)^{-\tau} (1 + \varepsilon \varsigma^{-1} |T|_{s_2}) |T|_{s_2} \|\mathcal{J}^{-1}w\|_{s_2} \\
&\leq c(\gamma, \gamma_1, \varsigma, \tau, s_2) (s_2 - s_1)^{-\tau} (s_4 - s_3)^{-\tau} N^{\tau+\kappa_0} \\
&\quad \times (1 + \varepsilon \varsigma^{-1} |T|_{s_2}) |T|_{s_2} (\|w\|_{s_3} + \varepsilon |T|_{s_2} \|w\|_{s_4}), \quad (2.39)
\end{aligned}$$

$$\|\mathcal{J}^{-1}J_R^S J_R^{-1}w_R\|_{s_1} \leq c(\varsigma, \tau, s_1, \gamma_1, \gamma) N^{\tau+\kappa_0} (s_3 - s_2)^{-\tau}$$

$$\begin{aligned}
& \times (\|J_R^S J_R^{-1} w_R\|_{s_3} + \epsilon |T|_{s_1} \|J_R^S J_R^{-1} w_R\|_{s_2}) \\
& \leq c(\varsigma, \tau, s_1, s_2, s_3, \gamma_1, \gamma) N^{\tau+\kappa_0} (s_3 - s_2)^{-\tau} \\
& \quad \times (|T|_{s_3} \|J_R^{-1} w_R\|_{s_3} + \epsilon |T|_{s_1} |T|_{s_2} \|J_R^{-1} w_R\|_{s_2}) \\
& \leq c(\varsigma, \tau, s_1, s_2, s_3, \gamma_1, \gamma) N^{\tau+\kappa_0} (s_3 - s_2)^{-\tau} \\
& \quad \times (|T|_{s_3} (s_4 - s_3)^{-\tau} (1 + \epsilon \varsigma^{-1} |T|_{s_4}) \|w\|_{s_4} \\
& \quad + \epsilon |T|_{s_1} |T|_{s_2} (s_3 - s_2)^{-\tau} (1 + \epsilon \varsigma^{-1} |T|_{s_3}) \|w\|_{s_3}) \\
& \leq c(\varsigma, \tau, s_1, s_2, s_3, \gamma_1, \gamma) N^{\tau+\kappa_0} (s_3 - s_2)^{-\tau} |T|_{s_3} (1 + \epsilon \varsigma^{-1} |T|_{s_4}) \\
& \quad \times ((s_4 - s_3)^{-\tau} \|w\|_{s_4} + \epsilon |T|_{s_2} (s_3 - s_2)^{-\tau} \|w\|_{s_3}), \tag{2.40}
\end{aligned}$$

$$\begin{aligned}
\|J_R^{-1} J_S^R \mathcal{J}^{-1} J_R^S J_R^{-1} w_R\|_{s_1} & \leq c(\gamma, \tau, s_2) (s_2 - s_1)^{-\tau} (1 + \epsilon \varsigma^{-1} |T|_{s_2}) \|J_S^R \mathcal{J}^{-1} J_R^S J_R^{-1} w_R\|_{s_2} \\
& \leq c(\gamma, \tau, s_2) (s_2 - s_1)^{-\tau} (1 + \epsilon \varsigma^{-1} |T|_{s_2}) |T|_{s_2} \|\mathcal{J}^{-1} J_R^S J_R^{-1} w_R\|_{s_2} \\
& \leq c(\varsigma, \tau, s_1, s_2, s_3, \gamma_1, \gamma) N^{\tau+\kappa_0} (s_3 - s_2)^{-\tau} (s_2 - s_1)^{-\tau} |T|_{s_3}^2 \\
& \quad \times (1 + \epsilon \varsigma^{-1} |T|_{s_4})^2 ((s_4 - s_3)^{-\tau} \|w\|_{s_4} \\
& \quad + \epsilon |T|_{s_2} (s_3 - s_2)^{-\tau} \|w\|_{s_3}). \tag{2.41}
\end{aligned}$$

The terms $\|J_R^{-1} w_R\|_{s_1}$ and $\|\mathcal{J}^{-1} w_R\|_{s_1}$ can be controlled by using (2.27) and (2.33). Thus by (2.38)-(2.41), for $0 < s < \tilde{s}$, we conclude

$$\|(\mathcal{J}_\omega^{(N)})^{-1} w\|_s \leq c(\varsigma, \tau, s, \tilde{s}, \gamma_1, \gamma) N^{\tau+\kappa_0} (1 + \epsilon \varsigma^{-1} |T|_{\tilde{s}})^3 (\tilde{s} - s)^{-\tau} \|w\|_{\tilde{s}},$$

which together with Lemma 2.7 gives (2.12).

3 Solution of the range equation (1.10) and the bifurcation equation (1.12)

In this section, we give the proof of Theorem 1.1. An equivalent result is:

Proposition 3.1. *Assume that Σ is a regular timelike minimal surface in \mathbf{R}^{1+n} which is diffeomorphic to a torus \mathcal{T} . There exists a Sobolev regularity parametrization*

$$\begin{aligned}
\mathbf{T}^2 & \mapsto \Sigma, \\
(t, \theta) & \mapsto (t, x(t, \theta))
\end{aligned}$$

of Σ such that $x(t, \theta) = t + \theta + u(t + \theta, \omega t)$ is time quasi-periodic with frequency $(1, \omega)$ and periodic in s with periodic T , and

$$|x_\theta|^2 x_{tt} - 2\langle x_t, x_\theta \rangle x_{t\theta} + (|x_t|^2 - 1) x_{\theta\theta} = 0,$$

where $\omega \in \mathcal{D}_{\gamma, \tau}$ satisfies nonresonant condition (1.6) and (2.11), $\mathcal{D}_{\gamma, \tau} \subset [0, \epsilon_0] \times [\omega_1, \omega_2]$ denote a Cantor like set of Lebesgue measure $|\mathcal{D}_{\gamma, \tau}| \geq \epsilon_0 (|\omega_2 - \omega_1| - C\gamma)$.

Furthermore, let $\tau > 0$, $0 < \sigma_0 + 32(\tau + 2) < \bar{\sigma}$ and $\epsilon_0 > 0$. The Sobolev regularity solution of above equation gives rise to a global and local uniqueness Sobolev regular timelike torus \mathcal{T} in \mathbf{R}^{1+n} , and a map $x(\epsilon, \omega) \in \mathbf{C}^1([0, \epsilon_0] \times [\omega_1, \omega_2]; \mathbf{H}_{\bar{\sigma}})$ with $\|u(\epsilon, \omega)\|_{\bar{\sigma}} = O(\epsilon)$, $\|D_{\epsilon, \omega} u(\epsilon, \omega)\|_{\bar{\sigma}} \leq C\gamma^{-1}$.

The proof of Proposition 3.1 is based on constructing a suitable Nash-Moser iteration scheme. Due to the powerful Nash-Moser iteration scheme, we can solve both the range equation (1.10) and the bifurcation equation (1.12) by the same scheme. So we only show the existence of solution for the range equation (1.10). The dependence upon the parameter ω of the solution of (1.10), as is well known, is more delicate since it involves in the small divisors of ω_j : it is, however, standard to check that this dependence is \mathbf{C}^1 on a bounded set of Diophantine numbers, for more details, see, for example, [4, 5].

We construct the “first step approximation”.

Lemma 3.1. *Assume that ω satisfies (1.6) and (2.11). Then system (2.2) has the “first step approximation” $w^1 \in \mathbf{H}_s^{(N_1)}$:*

$$\begin{aligned} w^1 = & 2\omega^2 \epsilon (I - \Psi^{(N_0)}) \Psi^{(N_1)} \Pi_{\mathbf{W}} \left(\left(\sum_{k=1}^n (v_{ky} + w_{ky}^0) + \frac{\epsilon}{2} |v_y + w_y^0|^2 \right) (v_{tt} + w_{tt}^0) \right. \\ & \left. - \left(\sum_{k=1}^n (v_{kt} + w_{kt}^0) + \epsilon \langle v_t + w_t^0, v_y + w_y^0 \rangle \right) (v_{ty} + w_{ty}^0) + \frac{\epsilon}{2} |v_t + w_t^0|^2 (v_{yy} + w_{yy}^0) \right), \end{aligned} \quad (3.1)$$

$$\begin{aligned} E^1 = & R^0 = 2\omega^2 \epsilon \Psi^{(N_1)} \left(\left(\sum_{k=1}^n w_{ky}^1 \right) w_{tt}^1 + \frac{\epsilon}{2} (2 \langle v_y + w_y^0, w_y^1 \rangle + |w_y^1|^2) w_{tt}^1 + \epsilon |w_y^1|^2 (v_{tt} + w_{tt}^0) \right. \\ & + \left(\sum_{k=1}^n w_{kt}^1 \right) w_{ty}^1 + \epsilon \left((\langle v_t + w_t^0, w_y^1 \rangle + \langle v_y + w_y^0, w_t^1 \rangle) w_{ty}^1 + \langle w_t^1, w_y^1 \rangle (v_{ty} + w_{ty}^0) \right) \\ & \left. + 2\epsilon (\langle v_t + w_t^0, w_t^1 \rangle + |w_t^1|^2) w_{yy}^1 \right). \end{aligned} \quad (3.2)$$

Proof. Assume that we have chosen suitable the “0th step” approximation solution w^0 . (In fact, we can choose $w^0 = 0$). Then the target is to get the “1th step” approximation solution.

Denote

$$E^0 = \mathcal{J}_\omega w^0 + 2\omega^2 \epsilon \Psi^{(N_1)} f(v_t, w_t^0, v_y, w_y^0, v_{tt}, w_{tt}^0, v_{yy}, w_{yy}^0). \quad (3.3)$$

By (2.2), we have

$$\begin{aligned} \mathcal{J}(w^0 + w^1) &= \mathcal{J}_\omega(w^0 + w^1) + 2\omega^2 \epsilon \Psi^{(N_1)} f(v_t, w_t^0 + w_t^1, v_y, w_y^0 + w_y^1, v_{tt}, w_{tt}^0 + w_{tt}^1, v_{yy}, w_{yy}^0 + w_{yy}^1) \\ &= \mathcal{J}_\omega w^0 + 2\omega^2 \epsilon \Psi^{(N_1)} f(v_t, w_t^0, v_y, v_y^0, v_{tt}, w_{tt}^0, v_{yy}, w_{yy}^0) + \mathcal{J}_\omega w^1 \\ &\quad + 2\omega^2 \epsilon \Psi^{(N_1)} D_w f(v_t, w_t^0, v_y, v_y^0, v_{tt}, w_{tt}^0, v_{yy}, w_{yy}^0) w^1 \\ &\quad + 2\omega^2 \epsilon \Psi^{(N_1)} (f(v_t, w_t^0 + w_t^1, v_y, w_y^0 + w_y^1, v_{tt}, w_{tt}^0 + w_{tt}^1, v_{yy}, w_{yy}^0 + w_{yy}^1) \\ &\quad - f(v_t, w_t^0, v_y, v_y^0, v_{tt}, w_{tt}^0, v_{yy}, w_{yy}^0) - D_w f(v_t, w_t^0, v_y, v_y^0, v_{tt}, w_{tt}^0, v_{yy}, w_{yy}^0) w^1) \\ &= E^0 + \mathcal{J}_\omega^{(N_1)} w^1 + R^0, \end{aligned} \quad (3.4)$$

where

$$\begin{aligned} \mathcal{J}_\omega^{(N_1)} w^1 &= \mathcal{J}_\omega w^1 + 2\omega^2 \epsilon \Psi^{(N_1)} D_w f(v_t, w_t^0, v_{tt}, w_{tt}^0) w^1 \\ &= \mathcal{J}_\omega w^1 + 2\omega^2 \epsilon \Psi^{(N_1)} \left(\left(\sum_{k=1}^n (v_{ky} + w_{ky}^0) + \frac{\epsilon}{2} |v_y + w_y^0|^2 \right) w_{tt}^1 \right. \end{aligned}$$

$$\begin{aligned}
& -(\sum_{k=1}^n (v_{ky} + w_{ky}^0) + \epsilon \langle v_t + w_t^0, v_y + w_y^0 \rangle) w_{ty}^1 \\
& + \frac{\epsilon}{2} |v_t + w_t^0|^2 w_{yy}^1 + \sum_{k=1}^n w_{ky}^1 (v_{tt} + w_{tt}^0) + 2\epsilon \langle v_y + w_y^0, w_y^1 \rangle (v_{tt} + w_{tt}^0) \\
& - \sum_{k=1}^n w_{kt}^1 (v_{ty} + w_{ty}^0) - \epsilon (\langle v_t + w_t^0, w_y^1 \rangle + \langle w_t^1, v_y + w_y^0 \rangle) (v_{ty} + w_{ty}^0) \\
& + \epsilon \langle v_t + w_t^0, w_t^1 \rangle (v_{yy} + w_{yy}^0),
\end{aligned}$$

$$\begin{aligned}
R^0 &= 2\omega^2 \epsilon \Psi^{(N_1)} (f(v_t, w_t^0 + w_t^1, v_y, w_y^0 + w_y^1, v_{tt}, w_{tt}^0 + w_{tt}^1, v_{yy}, w_{yy}^0 + w_{yy}^1) \\
& \quad - f(v_t, w_t^0, v_y, v_y^0, v_{tt}, w_{tt}^0, v_{yy}, w_{yy}^0) - D_w f(v_t, w_t^0, v_y, v_y^0, v_{tt}, w_{tt}^0, v_{yy}, w_{yy}^0) w^1) \\
&= 2\omega^2 \epsilon \Psi^{(N_1)} ((\sum_{k=1}^n w_{ky}^1) w_{tt}^1 + \frac{\epsilon}{2} (2\langle v_y + w_y^0, w_y^1 \rangle + |w_y^1|^2) w_{tt}^1 + \epsilon |w_y^1|^2 (v_{tt} + w_{tt}^0) \\
& \quad + (\sum_{k=1}^n w_{kt}^1) w_{ty}^1 + \epsilon ((\langle v_t + w_t^0, w_y^1 \rangle + \langle v_y + w_y^0, w_t^1 \rangle) w_{ty}^1 + \langle w_t^1, w_y^1 \rangle (v_{ty} + w_{ty}^0)) \\
& \quad + 2\epsilon (\langle v_t + w_t^0, w_t^1 \rangle + |w_t^1|^2) w_{yy}^1).
\end{aligned}$$

Then taking

$$w^1 = -(\mathcal{J}_\omega^{(N_1)})^{-1} E^0 \in \mathbf{H}_s^{(N_1)},$$

thus it has

$$E^0 + \mathcal{J}_\omega^{(N_1)} w^1 = 0.$$

By (3.4), we deduce

$$\begin{aligned}
E^1 &:= R^0 = \mathcal{J}(w^0 + w^1) \\
&= 2\omega^2 \epsilon \Psi^{(N_1)} (f(v_t, w_t^0 + w_t^1, v_y, w_y^0 + w_y^1, v_{tt}, w_{tt}^0 + w_{tt}^1, v_{yy}, w_{yy}^0 + w_{yy}^1) \\
& \quad - f(v_t, w_t^0, v_y, v_y^0, v_{tt}, w_{tt}^0, v_{yy}, w_{yy}^0) - D_w f(v_t, w_t^0, v_y, v_y^0, v_{tt}, w_{tt}^0, v_{yy}, w_{yy}^0) w^1) \\
&= 2\omega^2 \epsilon \Psi^{(N_1)} ((\sum_{k=1}^n w_{ky}^1) w_{tt}^1 + \frac{\epsilon}{2} (2\langle v_y + w_y^0, w_y^1 \rangle + |w_y^1|^2) w_{tt}^1 + \epsilon |w_y^1|^2 (v_{tt} + w_{tt}^0) \\
& \quad + (\sum_{k=1}^n w_{kt}^1) w_{ty}^1 + \epsilon ((\langle v_t + w_t^0, w_y^1 \rangle + \langle v_y + w_y^0, w_t^1 \rangle) w_{ty}^1 + \langle w_t^1, w_y^1 \rangle (v_{ty} + w_{ty}^0)) \\
& \quad + 2\epsilon (\langle v_t + w_t^0, w_t^1 \rangle + |w_t^1|^2) w_{yy}^1).
\end{aligned}$$

In fact, by (2.2) and (3.3), we can obtain

$$E^0 = 2\omega^2 \epsilon (I - \Psi^{(N_0)}) \Psi^{(N_1)} f(v_t, w_t^0, v_y, v_y^0, v_{tt}, w_{tt}^0, v_{yy}, w_{yy}^0). \quad (3.5)$$

This completes the proof. \square

In order to prove the convergence of the Newton algorithm, the following estimate is needed. Firstly, for convenience, we define

$$\tilde{E}^0 := 2\omega^2 \epsilon \Psi^{(N_1)} f(v_t, w_t^0, v_y, v_y^0, v_{tt}, w_{tt}^0, v_{yy}, w_{yy}^0). \quad (3.6)$$

Lemma 3.2. *Assume that ω satisfies (1.6) and (2.11). Then for any $0 < \alpha < \sigma$, the following estimates hold:*

$$\begin{aligned} \|w^1\|_{\sigma-\alpha} &\leq 2\omega^2 \epsilon C(\alpha) C_{4\epsilon} N_1^{\tau+\kappa_0+4} (\|w^0\|_\sigma^3 + \|v\|_\sigma^3) \\ &\quad \times (1 + \epsilon \varsigma^{-1} C_{\epsilon,\sigma} N_1^4 (\|w^0\|_{\sigma+s'}^2 + \|v\|_{\sigma+s'}^2 + 2\|w^0\|_{\sigma+s'} + 2\|v\|_{\sigma+s'}))^3, \end{aligned}$$

$$\begin{aligned} \|E^1\|_{\sigma-\alpha} &\leq 8\omega^6 \epsilon^3 N_1^{12+2(\tau+\kappa_0)} C_{2\epsilon} C^2(\alpha) (C_3 + \|v\|_{\sigma-\alpha} + \|w^0\|_{\sigma-\alpha}) (\|v\|_\sigma^3 + \|w^0\|_\sigma^3)^2 \\ &\quad \times (1 + \epsilon \varsigma^{-1} C_{\epsilon,\sigma} N_1^4 (\|w^0\|_{\sigma+s'}^2 + \|v\|_{\sigma+s'}^2 + 2\|w^0\|_{\sigma+s'} + 2\|v\|_{\sigma+s'}))^6 \\ &\quad \times [1 + C(\alpha) (1 + 2\omega \epsilon^2 \varsigma^{-1} C'_{\epsilon,\sigma} N_1^{8+\tau+\kappa_0} (\|w^0\|_{\sigma+s'}^2 + \|v\|_{\sigma+s'}^2 + 2\|w^0\|_{\sigma+s'} + 2\|v\|_{\sigma+s'}))^3 \\ &\quad \times (\|v\|_\sigma^3 + \|w^0\|_\sigma^3)], \end{aligned} \quad (3.7)$$

where $C(\alpha)$ is defined in (3.8), s' is a constant satisfying $s \geq \frac{7}{2}$, $C_{4\epsilon}$, $C_{\epsilon,\sigma}$ and C_ϵ denote constant depending on ϵ and σ , respectively.

Proof. Denote

$$C(\alpha) = c(\varsigma, \tau, \sigma, \gamma_1, \gamma) \alpha^{-\tau}. \quad (3.8)$$

From the definition of w^1 in (3.1), by Lemma 2.2, (2.1), (3.6) and (2.8), we derive

$$\begin{aligned} \|w^1\|_{\sigma-\alpha} &= \| -(\mathcal{J}_\omega^{(N_1)})^{-1} E^0 \|_{\sigma-\alpha} \\ &\leq C(\alpha) N_1^{\tau+\kappa_0} (1 + \epsilon \varsigma^{-1} C_{\epsilon,\sigma} N_1^4 (\|w^0\|_{\sigma+s'}^2 + \|v\|_{\sigma+s'}^2 + 2\|w^0\|_{\sigma+s'} + 2\|v\|_{\sigma+s'}))^3 \|E^0\|_\sigma \\ &\leq 2\omega^2 \epsilon C(\alpha) (1 + \epsilon \varsigma^{-1} C_{\epsilon,\sigma} N_1^4 (\|w^0\|_{\sigma+s'}^2 + \|v\|_{\sigma+s'}^2 + 2\|w^0\|_{\sigma+s'} + 2\|v\|_{\sigma+s'}))^3 \\ &\quad \times \|\Psi^{(N_1)} f(v_t, w_t^0, v_y, v_y^0, v_{tt}, w_{tt}^0, v_{yy}, w_{yy}^0)\|_{\sigma+\tau+\kappa_0} \\ &\leq 2\omega^2 \epsilon C(\alpha) C_{4\epsilon} N_1^{\tau+\kappa_0+4} (\|w^0\|_\sigma^3 + \|v\|_\sigma^3) \\ &\quad \times (1 + \epsilon \varsigma^{-1} C_{\epsilon,\sigma} N_1^4 (\|w^0\|_{\sigma+s'}^2 + \|v\|_{\sigma+s'}^2 + 2\|w^0\|_{\sigma+s'} + 2\|v\|_{\sigma+s'}))^3. \end{aligned} \quad (3.9)$$

By (2.1), (2.7), (3.9) and the definition of E^1 , we have

$$\begin{aligned} \|E^1\|_{\sigma-\alpha} &= \|\Psi^{(N_1)} (f(v_t, w_t^0 + w_t^1, v_y, w_y^0 + w_y^1, v_{tt}, w_{tt}^0 + w_{tt}^1, v_{yy}, w_{yy}^0 + w_{yy}^1) \\ &\quad - f(v_t, w_t^0, v_y, v_y^0, v_{tt}, w_{tt}^0, v_{yy}, w_{yy}^0) - D_w f(v_t, w_t^0, v_y, v_y^0, v_{tt}, w_{tt}^0, v_{yy}, w_{yy}^0) w^1) \|_{\sigma-\alpha} \\ &\leq 2\omega^2 \epsilon C_{2\epsilon} N_1^4 (C_3 + \|w^0\|_{\sigma-\alpha} + \|v^0\|_{\sigma-\alpha}) (\|w^1\|_{\sigma-\alpha}^2 + \|w^1\|_{\sigma-\alpha}^3) \\ &= 2\omega^2 \epsilon C_{2\epsilon} N_1^4 (C_3 + \|w^0\|_{\sigma-\alpha} + \|v^0\|_{\sigma-\alpha}) \|w^1\|_{\sigma-\alpha}^2 (1 + \|w^1\|_{\sigma-\alpha}) \\ &\leq 2\omega^2 \epsilon C_{2\epsilon} C^2(\alpha) N_1^{4+2(\tau+\kappa_0)} (C_3 + \|v\|_{\sigma-\alpha} + \|w^0\|_{\sigma-\alpha}) \\ &\quad \times (1 + \epsilon \varsigma^{-1} C_{\epsilon,\sigma} N_1^4 (\|w^0\|_{\sigma+s'}^2 + \|v\|_{\sigma+s'}^2 + 2\|w^0\|_{\sigma+s'} + 2\|v\|_{\sigma+s'}))^6 \|E^0\|_\sigma^2 \\ &\quad \times (1 + C(\alpha) N_1^{\tau+\kappa_0} (1 + \epsilon \varsigma^{-1} C_{\epsilon,\sigma} N_1^4 (\|w^0\|_{\sigma+s'}^2 + \|v\|_{\sigma+s'}^2 + 2\|w^0\|_{\sigma+s'} + 2\|v\|_{\sigma+s'}))^3 \|E^0\|_\sigma) \end{aligned}$$

$$\begin{aligned}
&\leq 2\omega^2 \epsilon C_{2\epsilon} C^2(\alpha) (C_3 + \|v\|_{\sigma-\alpha} + \|w^0\|_{\sigma-\alpha}) \|\tilde{E}^0\|_{\sigma+2+(\tau+\kappa_0)}^2 \\
&\quad \times (1 + \epsilon \varsigma^{-1} C_{\epsilon,\sigma} N_1^4 (\|w^0\|_{\sigma+s'}^2 + \|v\|_{\sigma+s'}^2 + 2\|w^0\|_{\sigma+s'} + 2\|v\|_{\sigma+s'}))^6 \\
&\quad \times \left(1 + C(\alpha) (1 + \epsilon \varsigma^{-1} C_{\epsilon,\sigma} N_1^4 (\|w^0\|_{\sigma+s'}^2 + \|v\|_{\sigma+s'}^2 + 2\|w^0\|_{\sigma+s'} + 2\|v\|_{\sigma+s'}))^3 \|\tilde{E}^0\|_{\sigma+\tau+\kappa_0}\right) \\
&\leq 8\omega^6 \epsilon^3 N_1^{12+2(\tau+\kappa_0)} C_{2\epsilon} C^2(\alpha) (C_3 + \|v\|_{\sigma-\alpha} + \|w^0\|_{\sigma-\alpha}) (\|v\|_{\sigma}^3 + \|w^0\|_{\sigma}^3)^2 \\
&\quad \times (1 + \epsilon \varsigma^{-1} C_{\epsilon,\sigma} N_1^4 (\|w^0\|_{\sigma+s'}^2 + \|v\|_{\sigma+s'}^2 + 2\|w^0\|_{\sigma+s'} + 2\|v\|_{\sigma+s'}))^6 \\
&\quad \times [1 + C(\alpha) (1 + 2\omega \epsilon^2 \varsigma^{-1} C'_{\epsilon,\sigma} N_1^{8+\tau+\kappa_0} (\|w^0\|_{\sigma+s'}^2 + \|v\|_{\sigma+s'}^2 + 2\|w^0\|_{\sigma+s'} + 2\|v\|_{\sigma+s'}))^3 \\
&\quad \times (\|v\|_{\sigma}^3 + \|w^0\|_{\sigma}^3)].
\end{aligned}$$

This completes the proof. \square

For $p \in \mathbf{N}$ and $0 < \sigma_0 < \bar{\sigma} < \sigma$, set

$$\sigma_p := \bar{\sigma} + \frac{\sigma - \bar{\sigma}}{2^p}, \quad (3.10)$$

$$\alpha_{p+1} := \sigma_p - \sigma_{p+1} = \frac{\sigma - \bar{\sigma}}{2^{p+1}}. \quad (3.11)$$

By (3.10)-(3.11), it follows that

$$\sigma_0 > \sigma_1 > \dots > \sigma_p > \sigma_{p+1} > \dots, \text{ for } p \in \mathbf{N}.$$

Define

$$\begin{aligned}
\mathcal{P}_1(w^0) &:= w^0 + w^1, \text{ for } w^0 \in \mathbf{H}_{\sigma_0}^{(N_0)}, \\
E^p &= \mathcal{J}\left(\sum_{k=0}^p w^k\right) = \mathcal{J}(\underbrace{\mathcal{P}_1 \circ \dots \circ \mathcal{P}_1}_{p})(w^0).
\end{aligned}$$

In fact, to obtain the “ p th” approximation solution $w^p \in \mathbf{H}_{\sigma_p}^{(N_p)}$ of system (2.2), by lemma 3.1, we need to solve following equations

$$\begin{aligned}
\mathcal{J}\left(\sum_{k=0}^p w^k\right) &= \mathcal{J}_{\omega}\left(\sum_{k=0}^p w^k\right) + 2\omega^2 \epsilon \Psi^{(N_p)} f\left(v_t, \sum_{k=0}^{p-1} w_t^k, v_y, \sum_{k=0}^{p-1} w_y^k, v_{tt}, \sum_{k=0}^{p-1} w_{tt}^k, v_{yy}, \sum_{k=0}^{p-1} w_{yy}^k\right) \\
&\quad + \mathcal{J}_{\omega} w^p + 2\omega^2 \epsilon \Psi^{(N_p)} D_w f\left(v_t, \sum_{k=0}^{p-1} w_t^k, v_y, \sum_{k=0}^{p-1} w_y^k, v_{tt}, \sum_{k=0}^{p-1} w_{tt}^k, v_{yy}, \sum_{k=0}^{p-1} w_{yy}^k\right) w^p \\
&\quad + 2\omega^2 \epsilon \Psi^{(N_p)} \left(f\left(v_t, \sum_{k=0}^p w_t^k, v_y, \sum_{k=0}^p w_y^k, v_{tt}, \sum_{k=0}^p w_{tt}^k, v_{yy}, \sum_{k=0}^p w_{yy}^k\right)\right. \\
&\quad \left.- f\left(v_t, \sum_{k=0}^{p-1} w_t^k, v_y, \sum_{k=0}^{p-1} w_y^k, v_{tt}, \sum_{k=0}^{p-1} w_{tt}^k, v_{yy}, \sum_{k=0}^{p-1} w_{yy}^k\right)\right) \\
&\quad \left.- D_w f\left(v_t, \sum_{k=0}^{p-1} w_t^k, v_y, \sum_{k=0}^{p-1} w_y^k, v_{tt}, \sum_{k=0}^{p-1} w_{tt}^k, v_{yy}, \sum_{k=0}^{p-1} w_{yy}^k\right)\right).
\end{aligned}$$

Then the ‘ p th’ step approximation $w^p \in \mathbf{H}_{\sigma_p}^{(N_p)}$:

$$w^p = -(\mathcal{J}_\omega^{(N_p)})^{-1} E^{p-1}, \quad (3.12)$$

where

$$\begin{aligned} E^p &= \sum_{k=0}^p w^k + 2\omega^2 \epsilon \Psi^{(N_p)} f(v_t, \sum_{k=0}^{p-1} w_t^k, v_y, \sum_{k=0}^{p-1} w_y^k, v_{tt}, \sum_{k=0}^{p-1} w_{tt}^k, v_{yy}, \sum_{k=0}^{p-1} w_{yy}^k) \\ &= 2\omega^2 \epsilon (I - \Psi^{(N_{p-1})}) \Psi^{(N_p)} f(v_t, \sum_{k=0}^{p-1} w_t^k, v_y, \sum_{k=0}^{p-1} w_y^k, v_{tt}, \sum_{k=0}^{p-1} w_{tt}^k, v_{yy}, \sum_{k=0}^{p-1} w_{yy}^k). \end{aligned}$$

As done in Lemma 3.1, it is easy to get that

$$\begin{aligned} E^p := R^{p-1} &= 2\omega^2 \epsilon \Psi^{(N_p)} \left(\left(\sum_{k=1}^n w_{ky}^p \right) w_{tt}^p + \frac{\epsilon}{2} (2\langle v_y + w_y^{p-1}, w_y^p \rangle + |w_y^p|^2) w_{tt}^p + \epsilon |w_y^p|^2 (v_{tt} + w_{tt}^{p-1}) \right. \\ &\quad \left. + \left(\sum_{k=1}^n w_{kt}^p \right) w_{ty}^p + \epsilon \left(\langle v_t + w_t^{p-1}, w_t^p \rangle + \langle v_y + w_y^{p-1}, w_t^p \rangle \right) w_{ty}^p + \langle w_t^p, w_y^p \rangle (v_{ty} + w_{ty}^{p-1}) \right) \\ &\quad + 2\epsilon (\langle v_t + w_t^{p-1}, w_t^p \rangle + |w_t^p|^2) w_{yy}^p, \end{aligned} \quad (3.13)$$

$$\tilde{E}^p = 2\omega^2 \epsilon \Psi^{(N_p)} f(v_t, \sum_{k=0}^{p-1} w_t^k, v_y, \sum_{k=0}^{p-1} w_y^k, v_{tt}, \sum_{k=0}^{p-1} w_{tt}^k, v_{yy}, \sum_{k=0}^{p-1} w_{yy}^k). \quad (3.14)$$

Hence we only need to estimate R^{p-1} to prove the convergence of algorithm.

Remark 3.1. By (3.12) and Lemma 3.5, we deduce that the solution of the range equation (1.12) w depending on the solution of the bifurcation equation v , and by Lemma 2.1, there exist a constant R such that

$$\|w\|_{\bar{\sigma}} \leq R(1 + \|v\|_{\bar{\sigma}}^3).$$

Since the existence of the range equation (1.10) by using the same Nash-Moser iteration scheme, we can obtain the similar result with (3.12) and Lemma 3.5. Thus it follows Lemma 2.1 and (3.12) that

$$\|v\|_{\bar{\sigma}} \leq R'(1 + \|w\|_{\bar{\sigma}}^3), \quad (3.15)$$

where R' is a constant.

By Lemmas 2.1-2.2 and (3.15), we directly have

Lemma 3.3. Assume that ω satisfies (1.6) and (2.11). For $s_2 > s_1 \geq 0$, the linearized operator $\mathcal{J}_\omega^{(N_p)}$ satisfies

$$\begin{aligned} &\|(\mathcal{J}_\omega^{(N_p)}(\epsilon, w))^{-1} h\|_{s_1} \\ &\leq C(s_2 - s_1) N_p^{\tau + \kappa_0} \left(1 + \epsilon \varsigma^{-1} C_{\epsilon, R'} K(s_2) N_p^4 (\|w\|_{s_2+s'}^6 + \|w\|_{s_2+s'}^3 + \|w\|_{s_2+s'}^2 + \|w\|_{s_2+s'}) \right)^3 \|h\|_{s_2}, \end{aligned} \quad (3.16)$$

where $C(s_2 - s_1) = c(s_2 - s_1)^{-\tau}$, $c = c(\varsigma, \tau, s_1, s_2, \gamma_1, \gamma)$ and $C_{\epsilon, R'}$ denote constant, s' is a constant satisfying $s' \geq \frac{7}{2}$.

Lemma 3.4. *For any $s > 0$, there exist constants $C_{2\epsilon}$, $C_{3,R'}$ and $C_{4\epsilon,R'}$ such that*

$$\begin{aligned} & \|\Psi^{(N_p)}(f(v_t, w_t + h_t, v_y, w_y + h_y, v_{tt}, w_{tt} + h_{tt}, v_{yy}, w_{yy} + h_{yy}) \\ & \quad - f(v_t, w_t, v_y, w_y, v_{tt}, w_{tt}, v_{yy}, w_{yy}) - D_w f(v_t, w_t, v_y, w_y, v_{tt}, w_{tt}, v_{yy}, w_{yy})h)\|_s \\ & \leq C_{2\epsilon} N_p^4 (C_{3,R'} + \|w\|_s^3 + \|w\|_s)(\|h\|_s^2 + \|h\|_s^3), \end{aligned} \quad (3.17)$$

$$\|\Psi^{(N_p)} f(v_t, w_t, v_y, w_y, v_{tt}, w_{tt}, v_{yy}, w_{yy})\|_s \leq C_{4\epsilon,R'} N_p^4 (\|w\|_s^9 + \|w\|^6 + \|w\|_s^3). \quad (3.18)$$

The following result tells us that the existence of solutions for (2.2). A key point is to give a sufficient condition on the convergence of Newton algorithm.

Lemma 3.5. *Assume that ω satisfies (1.6) and (2.11). Then, for sufficiently small ϵ , equations (2.2) has a solution*

$$w^\infty = \sum_{k=0}^{\infty} w^k \in \mathbf{H}_{\bar{\sigma}}.$$

Proof. This proof is divided into the following situations. If

$$\epsilon \varsigma^{-1} C_{\epsilon,R'} K(\sigma_{p-1}) N_p^4 (\|w^{p-1}\|_{\sigma_{p-1}+s'}^6 + \|w^{p-1}\|_{\sigma_{p-1}+s'}^3 + \|w^{p-1}\|_{\sigma_{p-1}+s'}^2 + \|w^{p-1}\|_{\sigma_{p-1}+s'}) < 1,$$

we have

$$\begin{aligned} \text{Case 1 : } & 2C(\alpha_p) N_p^{\tau+\kappa_0} \|E^{p-1}\|_{\sigma_{p-1}} > 1, \|w^{p-1}\|_{\sigma_p} > \|w^{p-1}\|_{\sigma_p}^3 + C_{3,R'}, \|E^{p-2}\|_{\sigma_{p-2}} \leq \|E^{p-1}\|_{\sigma_{p-1}}, \\ \text{Case 2 : } & 2C(\alpha_p) N_p^{\tau+\kappa_0} \|E^{p-1}\|_{\sigma_{p-1}} > 1, \|w^{p-1}\|_{\sigma_p} > \|w^{p-1}\|_{\sigma_p}^3 + C_{3,R'}, \|E^{p-2}\|_{\sigma_{p-2}} > \|E^{p-1}\|_{\sigma_{p-1}}, \\ \text{Case 3 : } & 2C(\alpha_p) N_p^{\tau+\kappa_0} \|E^{p-1}\|_{\sigma_{p-1}} > 1, \|w^{p-1}\|_{\sigma_p} \leq \|w^{p-1}\|_{\sigma_p}^3 + C_{3,R'}, \|E^{p-2}\|_{\sigma_{p-2}} \leq \|E^{p-1}\|_{\sigma_{p-1}}, \\ \text{Case 4 : } & 2C(\alpha_p) N_p^{\tau+\kappa_0} \|E^{p-1}\|_{\sigma_{p-1}} > 1, \|w^{p-1}\|_{\sigma_p} \leq \|w^{p-1}\|_{\sigma_p}^3 + C_{3,R'}, \|E^{p-2}\|_{\sigma_{p-2}} > \|E^{p-1}\|_{\sigma_{p-1}}, \\ \text{Case 5 : } & 2C(\alpha_p) N_p^{\tau+\kappa_0} \|E^{p-1}\|_{\sigma_{p-1}} \leq 1, \|w^{p-1}\|_{\sigma_p} > \|w^{p-1}\|_{\sigma_p}^3 + C_{3,R'}, \|E^{p-2}\|_{\sigma_{p-2}} \leq \|E^{p-1}\|_{\sigma_{p-1}}, \\ \text{Case 6 : } & 2C(\alpha_p) N_p^{\tau+\kappa_0} \|E^{p-1}\|_{\sigma_{p-1}} \leq 1, \|w^{p-1}\|_{\sigma_p} > \|w^{p-1}\|_{\sigma_p}^3 + C_{3,R'}, \|E^{p-2}\|_{\sigma_{p-2}} > \|E^{p-1}\|_{\sigma_{p-1}}, \\ \text{Case 7 : } & 2C(\alpha_p) N_p^{\tau+\kappa_0} \|E^{p-1}\|_{\sigma_{p-1}} \leq 1, \|w^{p-1}\|_{\sigma_p} \leq \|w^{p-1}\|_{\sigma_p}^3 + C_{3,R'}, \|E^{p-2}\|_{\sigma_{p-2}} \leq \|E^{p-1}\|_{\sigma_{p-1}}, \\ \text{Case 8 : } & 2C(\alpha_p) N_p^{\tau+\kappa_0} \|E^{p-1}\|_{\sigma_{p-1}} \leq 1, \|w^{p-1}\|_{\sigma_p} \leq \|w^{p-1}\|_{\sigma_p}^3 + C_{3,R'}, \|E^{p-2}\|_{\sigma_{p-2}} > \|E^{p-1}\|_{\sigma_{p-1}}. \end{aligned}$$

We only prove the case 1, the rest case is the similar. By Lemma 3.3, (3.12) and (3.14), we derive

$$\begin{aligned} \|w^p\|_{\sigma_p} &= \| -(\mathcal{J}_\omega^{(N_p)})^{-1} E^{p-1} \|_{\sigma_p} \\ &\leq C(\alpha_p) N_p^{\tau+\kappa_0} \|E^{p-1}\|_{\sigma_{p-1}} (1 + \epsilon \varsigma^{-1} C_{\epsilon,R'} K(\sigma_{p-1}) N_p^4 (\|w^{p-1}\|_{\sigma_{p-1}+s'}^6 \\ &\quad + \|w^{p-1}\|_{\sigma_{p-1}+s'}^3 + \|w^{p-1}\|_{\sigma_{p-1}+s'}^2 + \|w^{p-1}\|_{\sigma_{p-1}+s'})) \\ &\leq 2C(\alpha_p) \|\tilde{E}^{p-1}\|_{\sigma_{p-1}+2(\tau+\kappa_0)}, \end{aligned} \quad (3.19)$$

where $c(\epsilon, \varsigma)$ is a constant depending on ϵ and ς , $K(\sigma_p)$ is a bounded constant due to (3.10).

Note that $N_p = N_0^p$, $\forall p \in \mathbf{N}$. By (3.13)-(3.19) and (3.17) in Lemma 3.4, we have

$$\|E^p\|_{\sigma_p} = 2\omega^2 \epsilon \|\Psi^{(N_p)}(f(v_t, \sum_{k=0}^p w_t^k, v_y, \sum_{k=0}^p w_y^k, v_{tt}, \sum_{k=0}^p w_{tt}^k, v_{yy}, \sum_{k=0}^p w_{yy}^k))\|_{\sigma_p}$$

$$\begin{aligned}
& -f(v_t, \sum_{k=0}^{p-1} w_t^k, v_y, \sum_{k=0}^{p-1} w_y^k, v_{tt}, \sum_{k=0}^{p-1} w_{tt}^k, v_{yy}, \sum_{k=0}^{p-1} w_{yy}^k) \\
& -D_w f(v_t, \sum_{k=0}^{p-1} w_t^k, v_y, \sum_{k=0}^{p-1} w_y^k, v_{tt}, \sum_{k=0}^{p-1} w_{tt}^k, v_{yy}, \sum_{k=0}^{p-1} w_{yy}^k) \|_{\sigma_p} \\
& \leq 2\omega^2 \epsilon C_{2\epsilon} N_p^4 (C_{3,R'} + \|w^{p-1}\|_{\sigma_p}^3 + \|w^{p-1}\|_{\sigma_p}) (\|w^p\|_{\sigma_p}^2 + \|w^p\|_{\sigma_p}^3) \\
& = 2\omega^2 \epsilon C_{2\epsilon} N_p^4 (C_{3,R'} + \|w^{p-1}\|_{\sigma_p}^3 + \|w^{p-1}\|_{\sigma_p}) \|w^p\|_{\sigma_p}^2 (1 + \|w^p\|_{\sigma_p}) \\
& \leq 2\omega^2 \epsilon C_{2\epsilon} C^2(\alpha_p) N_p^{2(\tau+\kappa_0+2)} \|E^{p-1}\|_{\sigma_{p-1}}^2 (C_{3,R'} + \|w^{p-1}\|_{\sigma_p}^3 + \|w^{p-1}\|_{\sigma_p}) \\
& \quad \times (1 + 2C(\alpha_p) N_p^{\tau+\kappa_0} \|E^{p-1}\|_{\sigma_{p-1}}) \\
& \leq 8C_\epsilon \omega^2 \epsilon C^4(\alpha_p) N_p^{4(\tau+\kappa_0+1)} \|E^{p-1}\|_{\sigma_{p-1}}^4 \\
& \leq (8C_\epsilon \omega^2 \epsilon)^{4+1} N_p^{(\tau+\kappa_0+1)4} N_{p-1}^{(\tau+\kappa_0+1)4^2} C^4(\alpha_p) C^{4^2}(\alpha_{p-1}) \|E^{p-2}\|_{\sigma_{p-2}}^{4^2} \\
& = (8C_\epsilon \omega^2 \epsilon)^{4+1} N_0^{4p(\tau+\kappa_0+1)+4^2(p-1)(\tau+\kappa_0+1)} C^4(\alpha_p) C^{4^2}(\alpha_{p-1}) \|E^{p-2}\|_{\sigma_{p-2}}^{4^2} \\
& \leq \dots \\
& \leq (8C_\epsilon \omega^2 \epsilon)^{\sum_{k=1}^{p-1} 4^k+1} N_0^{(\tau+\kappa_0+1)4^{p+2}} \|E^0\|_{\sigma_0}^{4^p} \prod_{k=1}^p C^{4^k}(\alpha_{p+1-k}) \\
& \leq (8C_\epsilon \omega^2 \epsilon)^{4^p} (N_0^{16(\tau+\kappa_0+1)} \|E^0\|_{\sigma_0})^{4^p} \prod_{k=1}^p C^{4^k}(\alpha_{p+1-k}) \\
& \leq (8C_\epsilon \omega^2 \epsilon)^{4^p} \|\tilde{E}^0\|_{\sigma_0+16(\tau+\kappa_0+1)}^{4^p} \prod_{k=1}^p C^{4^k}(\alpha_{p+1-k}) \\
& \leq (8^{4^2+1} C_\epsilon \omega^2 \epsilon c^{16}(\tau, \sigma, \bar{\sigma}, \gamma_1, \gamma)) \|\tilde{E}^0\|_{\sigma_0+16(\tau+\kappa_0+1)}^{4^p}. \tag{3.20}
\end{aligned}$$

Hence we can choose a small $\epsilon > 0$ such that

$$8^{4^2+1} C_\epsilon \omega^2 \epsilon c^{16}(\tau, \sigma, \bar{\sigma}, \gamma_1, \gamma) \|\tilde{E}^0\|_{\sigma_0+16(\tau+\kappa_0+1)}^{4^p} < 1.$$

We derive

$$\lim_{p \rightarrow \infty} \|E^p\|_{\sigma_p} = 0.$$

If

$$\epsilon \varsigma^{-1} C_{\epsilon, R'} K(\sigma_{p-1}) N_p^4 (\|w^{p-1}\|_{\sigma_{p-1}+s'}^6 + \|w^{p-1}\|_{\sigma_{p-1}+s'}^3 + \|w^{p-1}\|_{\sigma_{p-1}+s'}^2 + \|w^{p-1}\|_{\sigma_{p-1}+s'}) \geq 1,$$

we have

$$\begin{aligned}
& \text{Case 1}^* : \|w^{p-1}\|_{\sigma_{p-1}+s'} > 1, \\
& \text{Case 2}^* : 0 < \|w^{p-1}\|_{\sigma_{p-1}+s'} \leq 1.
\end{aligned}$$

For the case 1*, by Lemma 3.3, (3.12) and (3.14)-(3.15), we derive

$$\|w^p\|_{\sigma_p} = \| -(\mathcal{J}_\omega^{(N_p)})^{-1} E^{p-1} \|_{\sigma_p}$$

$$\begin{aligned}
&\leq C(\alpha_p)N_p^{\tau+\kappa_0}\|E^{p-1}\|_{\sigma_{p-1}}(1+\epsilon\varsigma^{-1}C_{\epsilon,R'}K(\sigma_{p-1})N_p^4(\|w^{p-1}\|_{\sigma_{p-1}+s'}^6 \\
&\quad +\|w^{p-1}\|_{\sigma_{p-1}+s'}^3+\|w^{p-1}\|_{\sigma_{p-1}+s'}^2+\|w^{p-1}\|_{\sigma_{p-1}+s'}))^3 \\
&\leq 2^4\epsilon^3\varsigma^{-3}C_{\epsilon,R'}^3C(\alpha_p)K^3(\sigma_{p-1})N_p^{\tau+\kappa_0+12}\|E^{p-1}\|_{\sigma_{p-1}} \\
&\quad \times(\|w^{p-1}\|_{\sigma_{p-1}+s'}^6+\|w^{p-1}\|_{\sigma_{p-1}+s'}^3+\|w^{p-1}\|_{\sigma_{p-1}+s'}^2+\|w^{p-1}\|_{\sigma_{p-1}+s'}))^3 \\
&\leq 1024\epsilon^3\varsigma^{-3}C_{\epsilon,R',\sigma,\bar{\sigma}}^3C(\alpha_p)N_p^{\tau+\kappa_0+2s'+12}\|w^{p-1}\|_{\sigma_{p-1}}^{18}\|E^{p-1}\|_{\sigma_{p-1}} \\
&\leq (1024\epsilon^3\varsigma^{-3}C_{\epsilon,R',\sigma,\bar{\sigma}}^3)^{18+1}C(\alpha_p)C^{18}(\alpha_{p-1})N_0^{(\tau+\kappa_0+2s'+12)p+18(\tau+\kappa_0+2s'+12)(p-1)} \\
&\quad \times\|w^{p-2}\|_{\sigma_{p-2}}^{18^2}\|E^{p-1}\|_{\sigma_{p-1}}\|E^{p-2}\|_{\sigma_{p-2}}^{18} \\
&\leq \dots \\
&\leq (1024\epsilon^3\varsigma^{-3}C_{\epsilon,R',\sigma,\bar{\sigma}}^3N_0^{\tau+\kappa_0+2s'+12})^{\sum_{k=0}^{p-1}18^k}\|w^0\|_{\sigma_0}^{18^p}\prod_{k=1}^p C^{18^{k-1}}(\alpha_{p+1-k})\|\tilde{E}^{p-k}\|_{\sigma_{p-k}+\tau+\kappa_0}^{18^{k-1}}, \quad (3.21)
\end{aligned}$$

where $C_{\epsilon,R',\sigma,\bar{\sigma}}$ denote a constant depending on $\epsilon, R', \sigma, \bar{\sigma}$.

But we will choose the initial step $w^0 = 0$ in this paper, which combining with (3.21) leads to $\|w^p\|_{\sigma_p} = 0, \forall p \in \mathbf{N}$. This contradicts with assumption $\|w\|_{\sigma_{p-1}+s'} > 1$. Hence, the case is not possible.

For the case 2*, we need to divide into the following situations

$$\begin{aligned}
\text{Case } 1^{**}: & 2C(\alpha_p)N_p^{\tau+\kappa_0}\|E^{p-1}\|_{\sigma_{p-1}} > 1, \|w^{p-1}\|_{\sigma_p} < \|w^{p-1}\|_{\sigma_p}^3 + C_{3,R'}, \|E^{p-2}\|_{\sigma_{p-2}} \leq \|E^{p-1}\|_{\sigma_{p-1}}, \\
\text{Case } 2^{**}: & 2C(\alpha_p)N_p^{\tau+\kappa_0}\|E^{p-1}\|_{\sigma_{p-1}} > 1, \|w^{p-1}\|_{\sigma_p} < \|w^{p-1}\|_{\sigma_p}^3 + C_{3,R'}, \|E^{p-2}\|_{\sigma_{p-2}} > \|E^{p-1}\|_{\sigma_{p-1}}, \\
\text{Case } 3^{**}: & 2C(\alpha_p)N_p^{\tau+\kappa_0}\|E^{p-1}\|_{\sigma_{p-1}} \leq 1, \|w^{p-1}\|_{\sigma_p} < \|w^{p-1}\|_{\sigma_p}^3 + C_{3,R'}, \|E^{p-2}\|_{\sigma_{p-2}} \leq \|E^{p-1}\|_{\sigma_{p-1}}, \\
\text{Case } 4^{**}: & 2C(\alpha_p)N_p^{\tau+\kappa_0}\|E^{p-1}\|_{\sigma_{p-1}} \leq 1, \|w^{p-1}\|_{\sigma_p} < \|w^{p-1}\|_{\sigma_p}^3 + C_{3,R'}, \|E^{p-2}\|_{\sigma_{p-2}} > \|E^{p-1}\|_{\sigma_{p-1}}, \\
\text{Case } 5^{**}: & 2C(\alpha_p)N_p^{\tau+\kappa_0}\|E^{p-1}\|_{\sigma_{p-1}} > 1, \|w^{p-1}\|_{\sigma_p} \geq \|w^{p-1}\|_{\sigma_p}^3 + C_{3,R'}, \|E^{p-2}\|_{\sigma_{p-2}} \leq \|E^{p-1}\|_{\sigma_{p-1}}, \\
\text{Case } 6^{**}: & 2C(\alpha_p)N_p^{\tau+\kappa_0}\|E^{p-1}\|_{\sigma_{p-1}} > 1, \|w^{p-1}\|_{\sigma_p} \geq \|w^{p-1}\|_{\sigma_p}^3 + C_{3,R'}, \|E^{p-2}\|_{\sigma_{p-2}} > \|E^{p-1}\|_{\sigma_{p-1}}, \\
\text{Case } 7^{**}: & 2C(\alpha_p)N_p^{\tau+\kappa_0}\|E^{p-1}\|_{\sigma_{p-1}} \leq 1, \|w^{p-1}\|_{\sigma_p} \geq \|w^{p-1}\|_{\sigma_p}^3 + C_{3,R'}, \|E^{p-2}\|_{\sigma_{p-2}} \leq \|E^{p-1}\|_{\sigma_{p-1}}, \\
\text{Case } 8^{**}: & 2C(\alpha_p)N_p^{\tau+\kappa_0}\|E^{p-1}\|_{\sigma_{p-1}} \leq 1, \|w^{p-1}\|_{\sigma_p} \geq \|w^{p-1}\|_{\sigma_p}^3 + C_{3,R'}, \|E^{p-2}\|_{\sigma_{p-2}} > \|E^{p-1}\|_{\sigma_{p-1}}.
\end{aligned}$$

Now we prove that the cases 5** – 8** are not possible. For simple, we only discuss the case 5**. By the similar process of (3.20), we have

$$\begin{aligned}
\|w^p\|_{\sigma_p} &= \| -(\mathcal{J}_\omega^{(N_p)})^{-1}E^{p-1}\|_{\sigma_p} \\
&\leq C(\alpha_p)N_p^{\tau+\kappa_0}\|E^{p-1}\|_{\sigma_{p-1}}(1+\epsilon\varsigma^{-1}C_{\epsilon,R'}K(\sigma_{p-1})N_p^4(\|w^{p-1}\|_{\sigma_{p-1}+s'}^6 \\
&\quad +\|w^{p-1}\|_{\sigma_{p-1}+s'}^3+\|w^{p-1}\|_{\sigma_{p-1}+s'}^2+\|w^{p-1}\|_{\sigma_{p-1}+s'}))^3 \\
&\leq 2^4\epsilon^3\varsigma^{-3}C_{\epsilon,\sigma,\bar{\sigma}}^3C(\alpha_p)N_p^{\tau+\kappa_0+12}\|E^{p-1}\|_{\sigma_{p-1}} \\
&\quad \times(\|w^{p-1}\|_{\sigma_{p-1}+s'}^6+\|w^{p-1}\|_{\sigma_{p-1}+s'}^3+\|w^{p-1}\|_{\sigma_{p-1}+s'}^2+\|w^{p-1}\|_{\sigma_{p-1}+s'}))^3 \\
&\leq 64\epsilon^3\varsigma^{-3}C_{\epsilon,\sigma,\bar{\sigma}}^3C(\alpha_p)N_p^{\tau+\kappa_0+s'+12}\|E^{p-1}\|_{\sigma_{p-1}} \\
&\leq 64\epsilon^3\varsigma^{-3}C_{\epsilon,\sigma,\bar{\sigma}}^32^{p\tau}N_0^{(\tau+\kappa_0+s'+12)p}(8^{4^2+1}C_\epsilon\omega^2\epsilon c^{16}(\tau, \sigma, \bar{\sigma}, \gamma_1, \gamma)\|\tilde{E}^0\|_{\sigma_0+16(\tau+\kappa_0+1)})^{4^{p-1}} \\
&\leq 64\epsilon^3\varsigma^{-3}C_{\epsilon,\sigma,\bar{\sigma}}^3(8^{4^2+1}2^\tau N_0^{(\tau+\kappa_0+s'+12)})C_\epsilon\omega^2\epsilon c^{16}(\tau, \sigma, \bar{\sigma}, \gamma_1, \gamma)\|\tilde{E}^0\|_{\sigma_0+16(\tau+\kappa_0+1)}^{4^{p-1}}, \quad (3.22)
\end{aligned}$$

where $C_{\epsilon,\sigma,\bar{\sigma}}^3$ is a constant depending on ϵ, σ and $\bar{\sigma}$.

We can choose sufficient small $\epsilon > 0$ such that

$$8^{4^2+1} 2^\tau N_0^{(\tau+\kappa_0+s'+12)} C_\epsilon \omega^2 \epsilon c^{16}(\tau, \sigma, \bar{\sigma}, \gamma_1, \gamma) \|\tilde{E}^0\|_{\sigma_0+16(\tau+\kappa_0+1)} < 1.$$

Thus, by (3.22), we have $\lim_{p \rightarrow \infty} \|w^p\|_{\sigma_p} = 0$. This contradicts $\|w^{p-1}\|_{\sigma_p} \geq 1 + C_3$. Hence, the cases $5^{**} - 8^{**}$ are not possible.

In the following, we only prove the case 1^{**} . The idea of proving the rest three cases is the same, so we omit it. By (3.17), (3.13)-(3.19) and (3.22), we derive

$$\begin{aligned} \|E^p\|_{\sigma_p} &= 2\omega^2 \epsilon \|\Psi^{(N_p)}(f(v_t, \sum_{k=0}^p w_t^k, v_y, \sum_{k=0}^p w_y^k, v_{tt}, \sum_{k=0}^p w_{tt}^k, v_{yy}, \sum_{k=0}^p w_{yy}^k) \\ &\quad - f(v_t, \sum_{k=0}^{p-1} w_t^k, v_y, \sum_{k=0}^{p-1} w_y^k, v_{tt}, \sum_{k=0}^{p-1} w_{tt}^k, v_{yy}, \sum_{k=0}^{p-1} w_{yy}^k) \\ &\quad - D_w f(v_t, \sum_{k=0}^{p-1} w_t^k, v_y, \sum_{k=0}^{p-1} w_y^k, v_{tt}, \sum_{k=0}^{p-1} w_{tt}^k, v_{yy}, \sum_{k=0}^{p-1} w_{yy}^k))\|_{\sigma_p} \\ &\leq 2\omega^2 \epsilon C_{2\epsilon} C^2(\alpha_p) N_p^{2(\tau+\kappa_0+2)} \|E^{p-1}\|_{\sigma_{p-1}}^2 (C_{3,R'} + \|w^{p-1}\|_{\sigma_p}^3 + \|w^{p-1}\|_{\sigma_p}) \\ &\quad \times (1 + 2C(\alpha_p) N_p^{\tau+\kappa_0} \|E^{p-1}\|_{\sigma_{p-1}}) \\ &\leq 4C_{2\epsilon,R'} \omega^2 \epsilon C^3(\alpha_p) N_p^{3(\tau+\kappa_0+1)} \|E^{p-1}\|_{\sigma_{p-1}}^3 \\ &\leq (4C_{2\epsilon,R'} \omega^2 \epsilon)^3 N_p^{(\tau+\kappa_0+1)^3} N_{p-1}^{(\tau+\kappa_0+1)^3} C^3(\alpha_p) C^{3^2}(\alpha_{p-1}) \|E^{p-2}\|_{\sigma_{p-2}}^{3^2} \\ &= (4C_{2\epsilon,R'} \omega^2 \epsilon)^{3+1} N_0^{3p(\tau+\kappa_0+1)+3^2(p-1)(\tau+\kappa_0+1)} C^3(\alpha_p) C^{3^2}(\alpha_{p-1}) \|E^{p-2}\|_{\sigma_{p-2}}^{3^2} \\ &\leq \dots \\ &\leq (4C_{2\epsilon,R'} \omega^2 \epsilon)^{\sum_{k=1}^{p-1} 3^k+1} N_0^{(\tau+\kappa_0+1)3^{p+2}} \|E^0\|_{\sigma_0}^{3^p} \prod_{k=1}^p C^{3^k}(\alpha_{p+1-k}) \\ &\leq (4C_{2\epsilon,R'} \omega^2 \epsilon)^{3^p} (N_0^{9(\tau+\kappa_0+1)} \|E^0\|_{\sigma_0})^{3^p} \prod_{k=1}^p C^{3^k}(\alpha_{p+1-k}) \\ &\leq (4C_{2\epsilon,R'} \omega^2 \epsilon)^{3^p} \|\tilde{E}^0\|_{\sigma_0+9(\tau+\kappa_0+1)}^{3^p} \prod_{k=1}^p C^{3^k}(\alpha_{p+1-k}) \\ &\leq (4C_{2\epsilon,R'} \omega^2 \epsilon c^{16}(\tau, \sigma, \bar{\sigma}, \gamma_1, \gamma) \|\tilde{E}^0\|_{\sigma_0+9(\tau+\kappa_0+1)})^{3^p}. \end{aligned} \tag{3.23}$$

Hence we choose a small $\epsilon > 0$ such that

$$4C_{2\epsilon,R'} \omega^2 \epsilon c^{16}(\tau, \sigma, \bar{\sigma}, \gamma_1, \gamma) \|\tilde{E}^0\|_{\sigma_0+9(\tau+\kappa_0+1)} < 1.$$

Then (3.22) implies that

$$\lim_{p \rightarrow \infty} \|E^p\|_{\sigma_p} = 0.$$

Therefore we conclude that (2.2) has a solution

$$w^\infty := \sum_{k=0}^{\infty} w^k \in \mathbf{H}_{\bar{\sigma}}.$$

This completes the proof. \square

Next result gives the local uniqueness of solutions for equation (2.2).

Lemma 3.6. *Assume that ω satisfies (1.6) and (2.11). Equation (2.2) has a unique solution $w \in \mathbf{H}_{\bar{\sigma}} \cap \mathbf{B}_1(0)$ obtained in Lemma 3.5.*

Proof. Let $w, w' \in \mathbf{H}_{\bar{\sigma}} \cap \mathbf{B}_1(0)$ be two solutions of system (2.2), where

$$\mathbf{B}_1(0) := \{w \mid \|w\|_s < \delta, \text{ for some } \delta < 1, \forall s > 16(\tau + \kappa_0 + 1)\}.$$

Write $\psi = w - w'$. Our target is to prove $\psi = 0$. By (2.2), we have

$$\begin{aligned} \mathcal{J}_\omega \psi &+ 2\omega^2 \epsilon \Psi^{(N_p)} D_w f(v_t, w'_t, v_y, w'_y, v_{tt}, w'_{tt}, v_{yy}, w'_{yy}) \psi \\ &+ 2\omega^2 \epsilon \Psi^{(N_p)} (f(v_t, w_t, v_y, w_y, v_{tt}, w_{tt}, v_{yy}, w_{yy}) - f(v_t, w'_t, v_y, w'_y, v_{tt}, w'_{tt}, v_{yy}, w'_{yy})) \\ &- D_w f(v_t, w'_t, v_y, w'_y, v_{tt}, w'_{tt}, v_{yy}, w'_{yy}) \psi = 0, \end{aligned}$$

which implies that

$$\begin{aligned} \psi &= -2\omega^2 \epsilon (\mathcal{J}_\omega + 2\omega^2 \epsilon \Psi^{(N_p)} D_w f(v_t, w'_t, v_y, w'_y, v_{tt}, w'_{tt}, v_{yy}, w'_{yy}))^{-1} \\ &\times \Psi^{(N_p)} (f(v_t, w_t, v_y, w_y, v_{tt}, w_{tt}, v_{yy}, w_{yy}) - f(v_t, w'_t, v_y, w'_y, v_{tt}, w'_{tt}, v_{yy}, w'_{yy})) \\ &- D_w f(v_t, w'_t, v_y, w'_y, v_{tt}, w'_{tt}, v_{yy}, w'_{yy}) \psi. \end{aligned} \quad (3.24)$$

If

$$\epsilon \varsigma^{-1} C_{\epsilon, R'} K(\sigma_{p-1}) N_p^4 (\|w'\|_{\sigma_{p-1}+s'}^6 + \|w'\|_{\sigma_{p-1}+s'}^3 + \|w'\|_{\sigma_{p-1}+s'}^2 + \|w'\|_{\sigma_{p-1}+s'}) < 1,$$

by Lemma 3.3, (3.17) in Lemma 3.4, (3.24) and $N_p = N_0^p, \forall p \in \mathbf{N}$, we have

$$\begin{aligned} \|\psi\|_{\sigma_p} &= 2\omega^2 \epsilon \|(\mathcal{J}_\omega^{N_p})^{-1} \Psi^{(N_p)} (f(v_t, w_t, v_y, w_y, v_{tt}, w_{tt}, v_{yy}, w_{yy}) \\ &- f(v_t, w'_t, v_y, w'_y, v_{tt}, w'_{tt}, v_{yy}, w'_{yy}) - D_w f(v_t, w'_t, v_y, w'_y, v_{tt}, w'_{tt}, v_{yy}, w'_{yy}) \psi)\|_{\sigma_p} \\ &\leq 2\omega^2 \epsilon C(\alpha_p) N_p^{\tau+\kappa_0+4} C_{2\epsilon} (C_{3, R'} + \|w\|_s^3 + \|w\|_s) (\|\psi\|_{\sigma_{p-1}}^2 + \|\psi\|_{\sigma_{p-1}}^3) \\ &\quad \times \left(1 + \epsilon \varsigma^{-1} C_{\epsilon, R'} K(\sigma_{p-1}) N_p^4 (\|w'\|_{\sigma_{p-1}+s'}^6 + \|w'\|_{\sigma_{p-1}+s'}^3 + \|w'\|_{\sigma_{p-1}+s'}^2 + \|w'\|_{\sigma_{p-1}+s'})\right)^3 \\ &\leq 8\omega^2 \epsilon C'(\varsigma, \delta, \epsilon, \sigma, \bar{\sigma}, \varsigma, R') N_p^{\tau+\kappa_0+8} \|\psi\|_{\sigma_{p-1}}^2 \\ &\leq (8\omega^2 \epsilon C'(\varsigma, \delta, \epsilon, \sigma, \bar{\sigma}, \varsigma, R'))^{2+1} N_p^{(\tau+\kappa_0+8)} N_{p-1}^{2(\tau+\kappa_0+8)} C(\alpha_p) C^2(\alpha_{p-1}) \|\psi\|_{\sigma_{p-2}}^2 \\ &\leq \dots \\ &\leq (8\omega^2 \epsilon C'(\varsigma, \delta, \epsilon, \sigma, \bar{\sigma}, \varsigma, R'))^{\sum_{k=0}^{p-1} 2^k} N_0^{(\tau+\kappa_0+8)(\sum_{k=0}^{p-1} 2^k)} \|\psi\|_{\sigma_0}^{2^p} \prod_{k=1}^p C^{2^{k-1}}(\alpha_{p+1-k}) \\ &\leq (8\omega^2 \epsilon C'(\varsigma, \delta, \epsilon, \sigma, \bar{\sigma}, \varsigma, R') N_0^{2(\tau+\kappa_0+8)}) \|\psi\|_{\sigma_0}^{2^p}. \end{aligned} \quad (3.25)$$

Choosing suitable $\epsilon < 1$ such that

$$8\omega^2 \epsilon C'(\varsigma, \delta, \epsilon, \sigma, \bar{\sigma}, \varsigma, R') N_0^{2(\tau+\kappa_0+8)} \|\psi\|_{\sigma_0} < 1.$$

For the case $\epsilon \varsigma^{-1} C_{\epsilon, R'} K(\sigma_{p-1}) N_p^4 (\|w'\|_{\sigma_{p-1}+s'}^6 + \|w'\|_{\sigma_{p-1}+s'}^3 + \|w'\|_{\sigma_{p-1}+s'}^2 + \|w'\|_{\sigma_{p-1}+s'}) \geq 1$, by the similar estimate (3.25), we can also prove the same result. Thus we conclude that

$$\lim_{p \rightarrow \infty} \|\psi\|_{\bar{\sigma}} = 0.$$

This completes the proof. \square

Acknowledgements The first author expresses his sincere thanks to Prof Gang Tian for suggesting this interesting problem and his suggestion and encouragement!

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